An Overview of Representation Theory, Lie Groups, and Lie Algebras

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1 Representation Theory fundamentals

Definition 1. A representation of a group G is a vector space V equipped with a group homomorphism from G to the general linear group of V, GL(V).

Theorem 1. *Every finite group has a representation.*

Proof. Take a finite group G of order n. Create a vector space $\mathbb{C}^n$. Note that we can map each of the vectors in its elementary basis $\mathcal{E}$ to group elements and vice versa. Specifically, define e_g to be the elementary basis vector corresponding to group element g.

We know that, in any finite group, left-multiplying every element of G with some $g \in G$ permutes the elements of G. We can define a similar action in $\mathbb{C}^n$ that transforms every basis vector $\vec{e_h}$ into another basis vector $\vec{e_{gh}}$ for all $h \in G$ and a fixed $g \in G$. Then, define the representation $(\mathbb{C}^n, \pi)$ with an isomorphism π such that $\pi(g)$ is the permutation matrix that results from applying g to any element in the group G. For example, take the group $G = C_3$. Then let $\vec{e_1}$ correspond to the identity e, $\vec{e_2}$ correspond to r, and $\vec{e_3}$ correspond to r^2. Then, we define π as follows: $\pi(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\pi(r) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $\pi(r^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. ■

Definition 2. Given a representation (V, π) of a group G, a subspace $W \in V$ is *invariant* iff, for all $\pi(g)$, $\pi(g)$ sends all elements of W to W.

From here, we can define subrepresentations, and then go on to define the notions of irreducible, reducible, and fully reducible.

Definition 3. Given a representation (V, π) of a group G, a subrepresentation (W, π) is a representation where W is invariant.

Because W is invariant, each of the operators $\pi(g)$ are still valid operators that performs actions only on W, which is what makes (W, π) a representation as well.

Every representation (V, π) has two subrepresentations – $(\{0\}, \pi)$ and (V, π). This is similar to how every linear transformation $T : V \to V$ has two obvious T-invariant subspaces, $\{0\}$ and V, except that this invariance is required of all $\pi(g)$ operators instead of a single operator.

For example, because all finite groups G are subgroups of a symmetric group S_n, Theorem 1 provides a representation for any group using permutation matrices. Permutation invariant functions of numbers, like sums, are preserved by such permutation matrices when they act on vectors with these lists of numbers. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

The sum of the components in the vector $(2, 3, 4)$ is preserved under this action.

Generalizing this idea, the set of vectors

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ such that } a + b + c = 0 \right\}$$

is a subspace of $\mathbb{C}^3$ and, being invariant under permutation matrices, is invariant for any representation that is defined in Theorem 1.

Definition 4. A representation (V, π) is irreducible iff it has no subrepresentation other than $(\{0\}, \pi)$ and (V, π).

Definition 5. A representation (V, π) is fully reducible iff V can be written as the direct sum of invariant subspaces $W_1, W_2, ...W_k$.

A fun fact: For any finite group G, every element has an orbit, meaning that $g^{|G|} = e$ for all $g \in G$. So, in a representation for G that is complex, $\pi(g)^{|G|}$ is I_V, or that $\pi(g)^{|G|} - I_V = 0$. Because the minimal polynomial of $\pi(g)$ must divide $x^{|G|} - 1$, and $x^{|G|} - I$ has distinct roots, the minimal polynomial must have distinct roots, so $\pi(g)$ must be diagonalizable. Additionally, the roots (or eigenvalues) must be among the roots of unity.

Extending from this is the fact that, if G is an abelian group, then all of $\pi(g)$ must be *simultaneously diagonalizable*. Thus, there exists a matrix representation of G with solely diagonal matrices in the same eigenbasis, whose eigenvalues are roots of unity. So, each eigenvector defines a 1-dimensional invariant subspace for V. Because V can be written as the direct sum of 1-dimensional invariant subspaces, **the finite group G itself can be written as the cartesian product of cyclic groups**.

Theorem 2 (Maschke's theorem)**.** *For all finite groups G, every representation (V, π) of G that is over $\mathbb{C}$ is fully reducible.*

I'll proceed to give an overview of the proof. It relies on an idea called Weyl's trick, which enables you to invent an inner product so that, for any complex representaiton (V, π) of a group G, $\pi(g)$ becomes unitary. From here, we can mimic the proof of the Spectral Theorem for Normal operators to show inductively that, if V has an invariant subspace W_1, then V can be the written as the direct sum of W_1 and another space W_2 that is orthogonal to W_1 with respect to the invented inner product. Thus, we can split V into a direct sum of invariant spaces $W_1, W_2, ..., W_k$, and thus a direct sum of irreducible representations.

Definition 6. Given two representations (V, π) and (W, ω) of the same group G, an intertwining operator L is a linear operator $L : V \to W$ such that $L \circ \pi(g) = \omega(g) \circ L$ for all $g \in G$.

The purpose of an intertwining operator is to create a notion of correspondence between two representations – that applying an operator in V and moving to W via L is the same as moving to W via L and then applying a similar operator in W. If L is an isomorphism, then the two representations in question are isomorphic, and one can be gotten from the other via conjugation, where $\omega(g) = L \circ \pi(g) \circ L^{-1}$.

Lemma 3 (Schur's lemma)**.** *Given two irreducible representations (V, π) and (W, ω) of G, an intertwining operator between them must either be trivial ($L\vec{v} = \vec{0}$ for all $\vec{v}$) or L must be an isomorphism.*

The intuition behind this is that, if an intertwining operator L was neither trivial nor an isomorphism, it must have a nontrivial kernel (not $\{0\}$ or V) or nontrival image (not $\{0\}$ or W). This kernel and image must be invariant in (V, π) and (W, ω), respectively. If these representations are irreducible, then the only possible invariant subspaces the kernel and image can be is $\{0\}$ or V, or $\{0\}$ or W, respectively, which is a contradiction. Hence, we've shown Schur's lemma.

2 Lie groups and Lie algebras

Lie groups are groups that also carry properties of calculus, such as being differentiable. For example, the set $\{z \in \mathbb{C}$ such that $|z| = 1\}$ forms a Lie group under complex multiplication. Each element can also be written in the form $e^{i\theta}$ by DeMoivre's theorem, which enables us to parametrize the Lie group in terms of θ and differentiate from there.

For every Lie group, a Lie algebra can be defined to capture the local behavior of a Lie group in the form of a vector space. For this example, the corresponding Lie algebra is $i\mathbb{R}$, which is the set of all possible values of the exponent $i\theta$ in our group above. $i\mathbb{R}$ forms a vector space via addition and scalar multiplication. As seen in this example, addition in a Lie algebra corresponds to composition (multiplication of complex numbers) in the Lie group, and scalar multiplication in a Lie algebra corresponds to exponentiation, or repeated composition, in the Lie group.

Definition 7. A Lie group is a *continuous* group described by several real parameters.

For example, $SO(3)$, the group of all 3-dimensional rotation operators, is a Lie group. If we formulate a rotation operator as an axis vector of length 1 combined with a rotation amount, $SO(3)$ can be visualized as a ball of radius π, where each point (x, y, z) has a direction that represents the axis of rotation, and a length $\sqrt{x^2 + y^2 + z^2}$ that represents the rotation amount.

In this paper, we will only be working with *matrix Lie groups*, which are closed under limits of sequences. This means that the limit of a sequence $(X_0, X_1, X_2, ...)$ must be in the group if invertible and not if not invertible.

Now we proceed to define what a Lie algebra is, which is an entity in its own right.

Definition 8. A Lie algebra is a vector space V endowed with a bilinear operator called a commutator $[X, Y] : V \times V \to V$ such that

1. $[X, Y] \in V$ (commutator is closed)

2. $[X, Y] = -[Y, X]$ (skew symmetric)

3. $[X, Y] = 0$ iff the Lie algebra is commutative

4. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

The following definition links Lie algebras with Lie groups, and we'll only be using Lie algebras in this context.

Definition 9. A Lie algebra of a matrix Lie group G is a vector space of matrices that contains an element X if and only if e^{tX} is in G for all real values of t.

The only bilinear form used for these kinds of Lie algebras is $[X, Y] = XY - YX$, but we won't be discussing these much in the paper.

The matrix exponent e^{tX} is defined using the Taylor series definition of e^x. It takes some rigor to define it and prove the convergence of such a series, which we won't delve into. In the end, for all matrices X, there exists an exponential e^{tX}, and there exists a logarithm $logX$ as well iff the Hilbert-Schmidt norm of $X - I_V$ is less than 1 or iff $X - I_V$ is nilpotent (some power of X becomes 0_V).

The interesting part is that, after going through additional rigors regarding exponentials of matrices, it can be shown that the set defined in Definition 9 forms a vector space, closed under addition and scalar multiplication. If X and Y are matrices such that e^{tX} and e^{tY} are in the Lie group for all $t \in \mathbb{R}$, then $X + Y$ is in the Lie algebra surprisingly because it turns out that $e^{t(X+Y)}$ is also in the corresponding Lie group for all t.

In addition, this kind of Lie algebra happens to be closed under conjugation. Given Q in Lie group G and X in Lie algebra G_A, $\pi(QXQ^{-1})$ is $\pi(Q)\pi(X)\pi(Q)^{-1}$.

Example 1. For example, the general linear group $GL(\mathbb{C}^n)$ is a Lie group, and its Lie algebra is the space of matrices $M_n(\mathbb{C})$. This is because, if $X \in M_n(\mathbb{C})$, then e^{tX} must be invertible for all $t \in \mathbb{R}$, so e^{tX} must be in $GL(\mathbb{C}^n)$. In the other direction, for any $e^{tX} \in GL(\mathbb{C}^n)$, its derivative evaluated at $t = 0$, which is $\frac{d(e^{tX})}{dt}_{t=0} = X$, must exist also and be in $M_n(\mathbb{C})$.

Example 2. A less obvious example is for the Lie group $SL(\mathbb{C}^n)$. The Lie algebra of this group is the vector space of matrices X such that $trace(X) = 0$. Denote this set as $sl(\mathbb{C}^n)$. Given $X \in M_n(\mathbb{C}^n)$, we use the fact that if $trace(X)$ is 0, then $det(e^{tX}) = 1$ for all t.[1] In the other direction, if $e^{tX} \in SL(\mathbb{C}^n)$ for all t, then $trace(X) = \frac{d(e^{tX})}{dt}_{t=0} = X$.

Now it's time to combine everything we've said about representations, Lie groups, and Lie algebras.

Definition 10. A representation (W, π) of a Lie group G is defined in the same way any representation is defined for any group: π is a homomorphism $G \to GL(W)$. On the other hand, a representation (V, ω) of a Lie algebra G_A is a homomorphism from G_A to $End(V)$ respecting addition and scalar multiplication, while also respecting the bilinear form (meaning that ω is a homomorphism $G_A \to End(V)$[2] with the additional property that, for all $X, Y \in G_A$, $\omega([X, Y]) = [\omega(X), \omega(Y)]$).

[1] This is simpler to prove for diagonalizable X. Say we can write X as BDB^{-1} where D is diagonal. Then $tr(D)$ is 0 as well. We can write e^{tX} as $Be^{tD}B^{-1}$, and e^{tD} would have to have a determinant of 1. As a consequence, e^{tX} would have a determinant of 1.

[2] The set of endomorphisms is not GL(W) because it can also include non-invertible matrices.

Example 3. The **standard representation** of a Lie algebra G_A is (G_A, π) where $\pi(X) = X$ for all $X \in G_A$. This particular representation, though, doesn't tell us anything new about a Lie algebra.

For Lie algebras, the same notions of invariance and reducibility carry over – if a representation of a Lie algebra (V, ω) is reducible, then there exists a nontrivial subspace of V invariant under all the operators $\pi(g)$ for all g in Lie algebra G_A.

Proposition 4.4 of Hall's *Lie Groups, Lie Algebras, and Representations* demonstrates that, given a representation (V, π) of a Lie group G, the corresponding Lie algebra G_A has a representation (V, ω) such that, for every element X in G_A, $\pi(e^X) = e^{\omega(X)}$ – the Lie group representation of e^X is the exponential of some Lie algebra representation of X. We can find the Lie algebra representation $\omega(X)$ by taking the derivative of $\pi(e^{tX})$ evaluated at $t = 0$.

Example 4. Let's try to find the representation of Lie group $SU(2)$. In this case, the vector space for our representation will be V_m, the set of polynomials of degree m with two complex variables, z_1 and z_2, or alternatively a two dimensional vector $\vec{z} \in \mathbb{C}^2$. Then we can define the representation (V_m, π) where $\pi(U)$ for $U \in SU(2)$ is defined as $[\pi(U)] \circ f(\vec{z}) = f(U^{-1}\vec{z})$. Note that the functions V_m form a vector space, even when the functions themselves aren't linear.

Example 5. Let's find *all* representations of $sl(\mathbb{C}^2)$, the Lie algebra of the special linear group of $\mathbb{C}^2$. This is the set of all 2x2 matrices with trace 0. To provide necessary background, we let the reader know that $su(2)$, the Lie algebra of $SU(2)$, is the set of all 2x2 skew-adjoint matrices with trace 0. Given that $su(2)$ is a vector space over $\mathbb{R}$ instead of $\mathbb{C}$, we also inform the reader that $sl(\mathbb{C}^2)$ can be obtained by allowing $su(2)$ to be a vector space over $\mathbb{C}$ by taking complex multiples and linear combinations of elements of $su(2)$. We can use the representation of $SU(2)$ to make a representation of $sl(\mathbb{C}^2)$ using the derivative trick. We have $\omega(X) = \frac{d\pi(e^{tX})}{dt}_{t=0}$, we get a representation (V_m, ω) for $su(2)$. In the same way we extended $su(2)$ to get $sl(\mathbb{C}^2)$, we can extend this representation to $sl(\mathbb{C}^2)$, and this representation behaves in the following ways on the basis $\{H, X, Y\}$ with $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

1. (V_m, ω) is irreducible, simply because V_m doesn't have any invariant subspaces under H, X, and Y.
2. Each monomial $z_1^k z_2^{m-k}$ is an eigenvector of $\omega(H)$, with eigenvalue $m - 2k$
3. $\omega(X)$ and $\omega(Y)$ change the k in the monomial $z_1^k z_2^{m-k}$ by -1 and $+1$ respectively.

In parallel, we can find the exact same structure for *any irreducible representation of* $sl(\mathbb{C}^2)$. For every irreducible representation (V, ω_V) of $sl(\mathbb{C}^2)$, there exists an integer $m \geq 0$ and a basis $\{u_0, u_1, ..., u_m\}$ such that

1. u_k is an eigenvector for $\omega_V(H)$ with eigenvalue $m - 2k$
2. $\omega_V(Y)$ moves u_k to u_{k+1} with some multiplying factor.
3. $\omega_V(X)$ moves u_k to u_{k-1} with some multiplying factor.

This gives rise to the fact that every irreducible representation of $sl(\mathbb{C}^2)$ of dimension $m+1$ is isomorphic to (V_m, ω). Using the additional fact that, because every representation (V, θ) of $sl(\mathbb{C}^2)$ is fully reducible, (V, θ)'s subrepresentations must have the same property, and so will (V, θ) itself as a whole.

Theorem 4. *(Theorem 4.34 of Hall)*

1. *Every eigenvalue of $\theta(H)$ is an integer.*
2. *If an integer k is an eigenvalue of $\theta(H)$, then so is $-|k|, -|k|+2, ..., |k|-2, |k|$.*
3. *$\theta(X)$ and $\theta(Y)$ are nilpotent.*

3 Citations

1. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, by Brian C. Hall
2. *Linear Algebra*, by Stephen H. Friedberg, Arnold J. lnsel, Lawrence E. Spence