Wythoff’s game

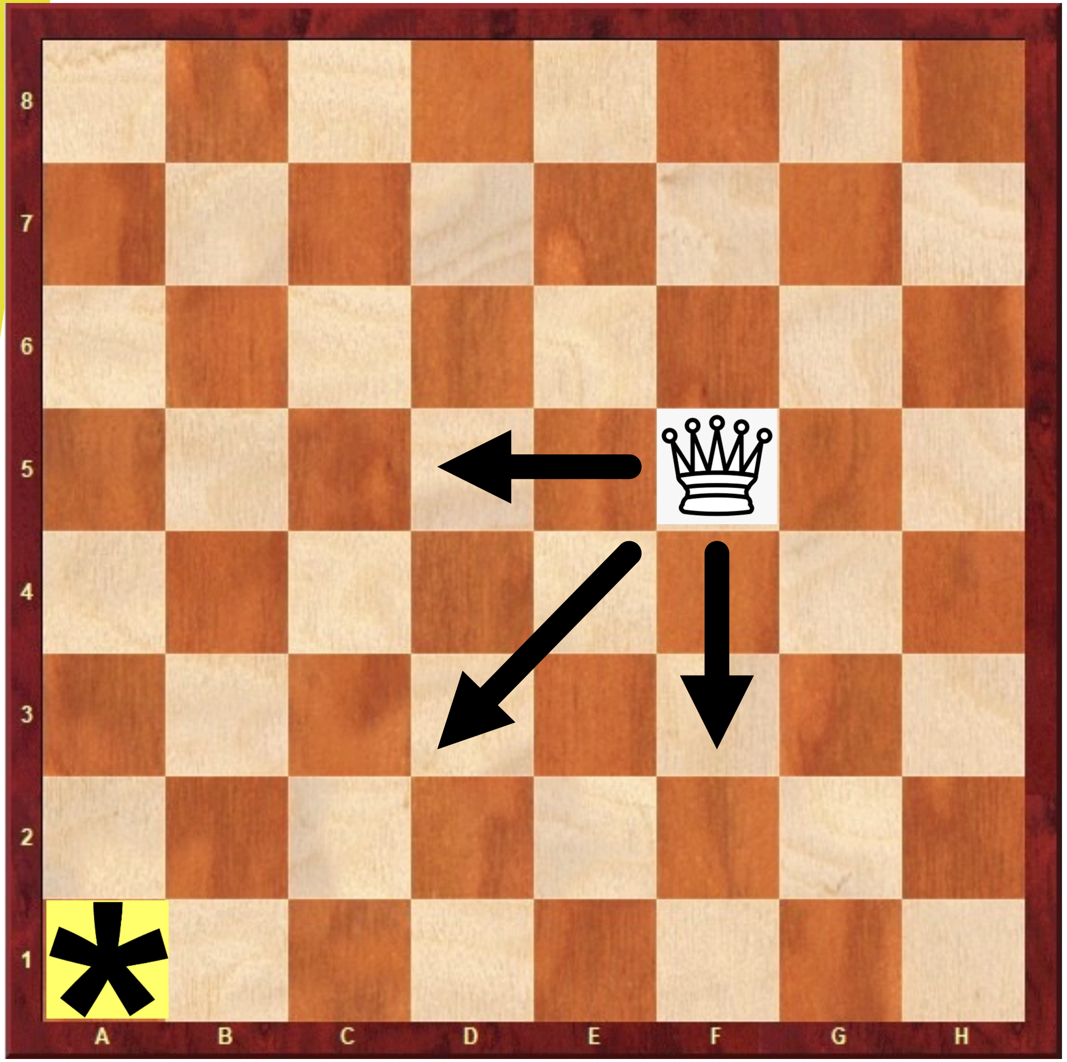
Ishwar Suriyaprakash

Game Rules

Wythoff's game is a 2-player game played on a chessboard.

The game starts with a queen placed at an arbitrary location.

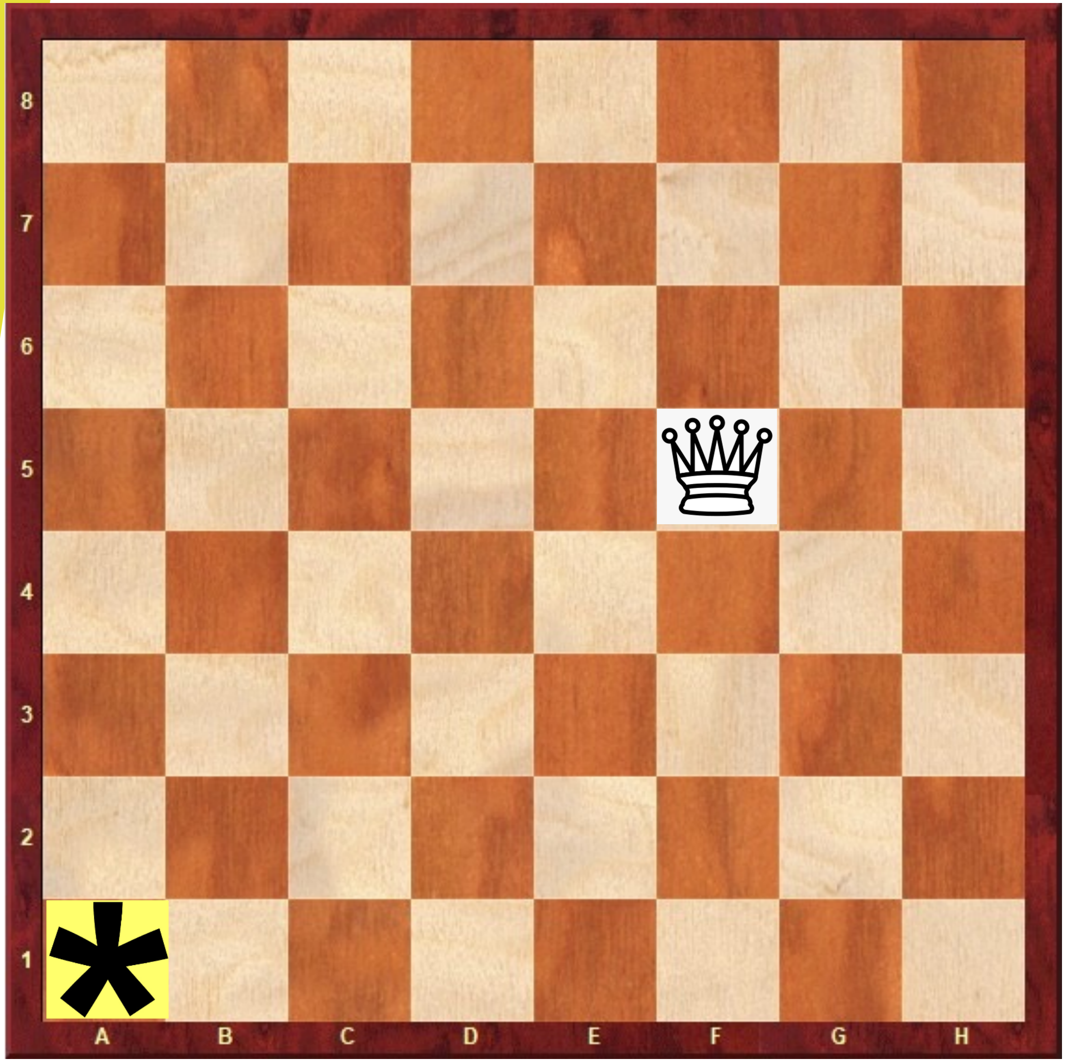
Game Rules



Each player takes turns moving the queen leftward, downward, or along the diagonal to the bottom-left by any number of squares.

The player who moves the queen to the star wins, or the player who is unable to move loses.

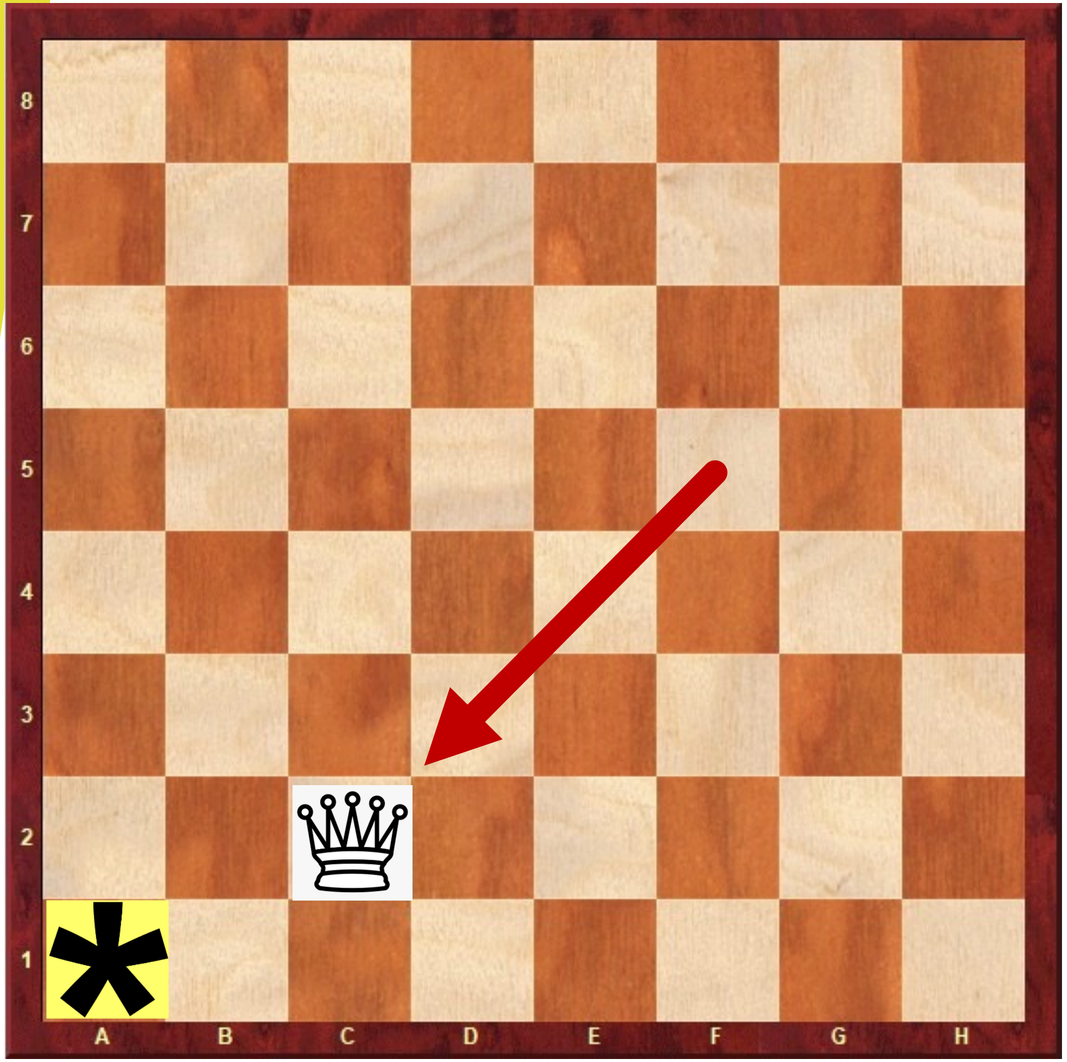
Game Rules



Example

Starts at F5

Game Rules

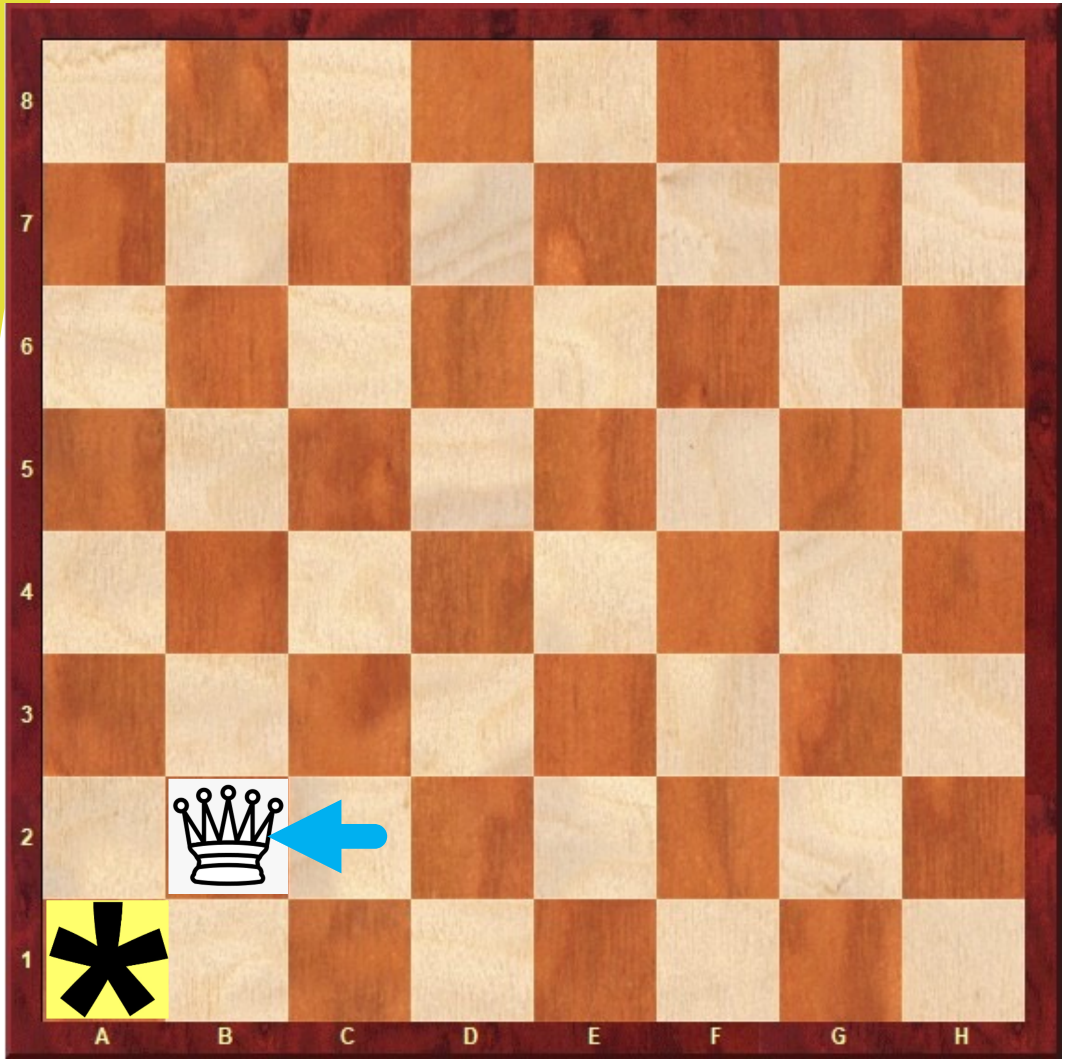


Example

Starts at F5

Player 1 moves to C2

Game Rules



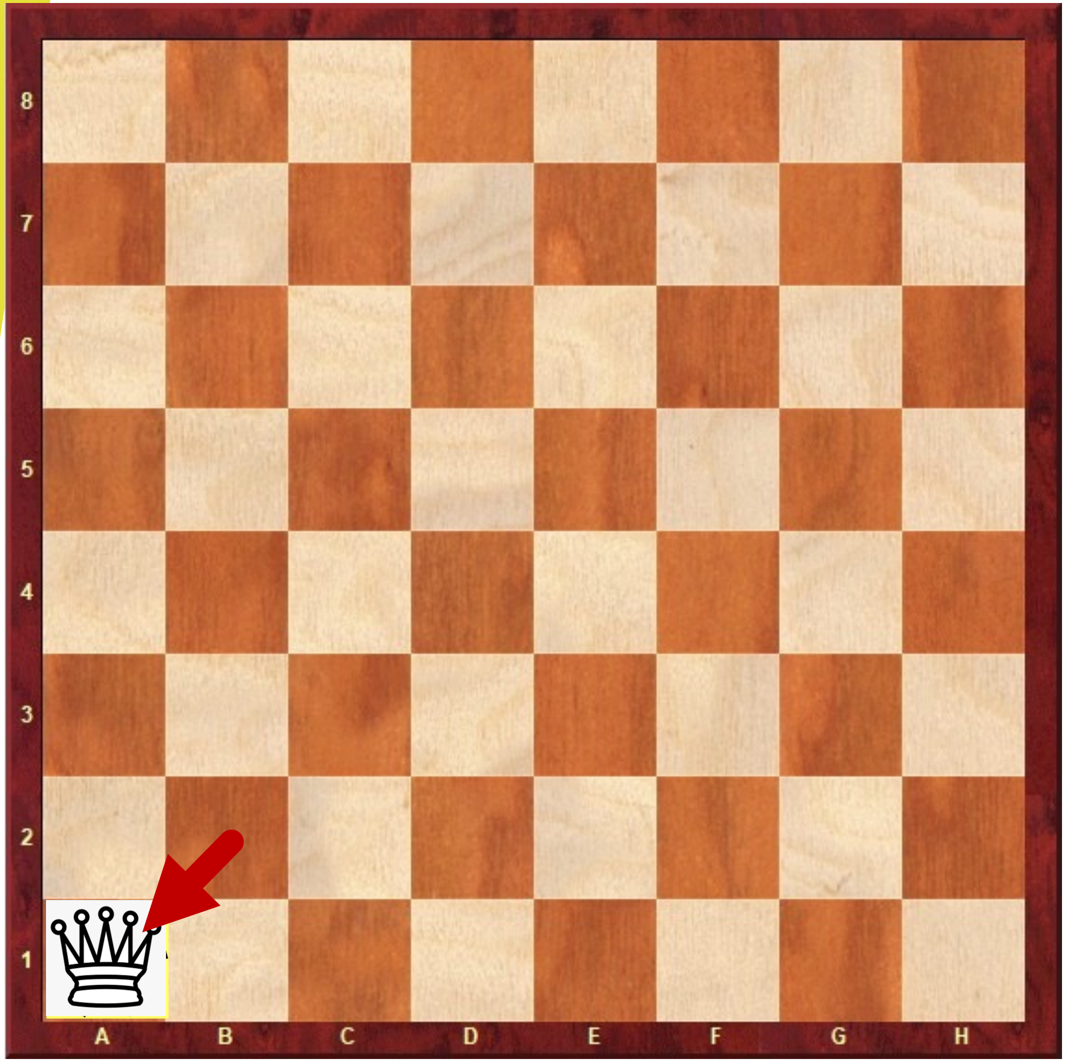
Example

Starts at F5

Player 1 moves to C2

Player 2 moves to B2

Game Rules



Example

Starts at F5

Player 1 moves to C2

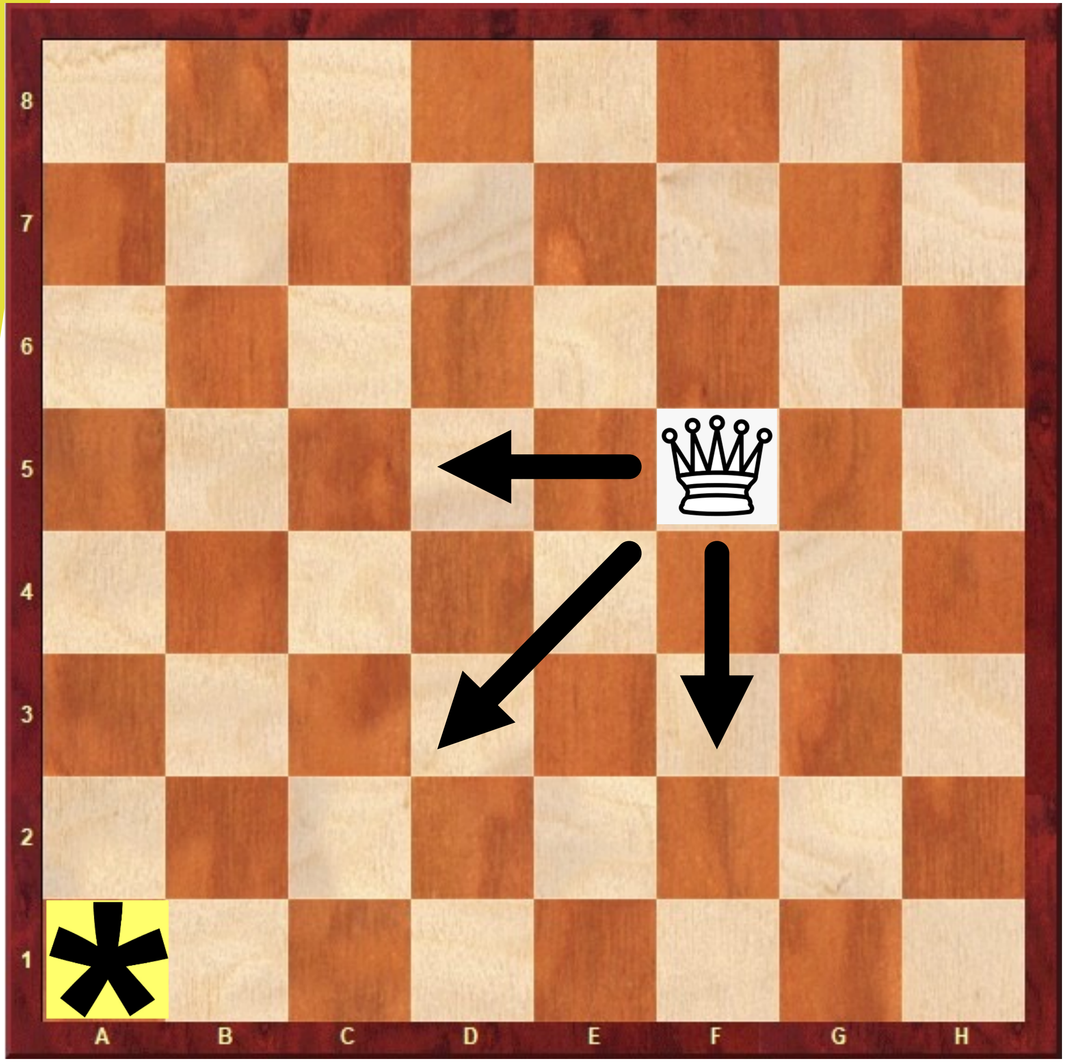
Player 2 moves to B2

Player 1 moves to A1 and wins!

Game Rules

Play time!

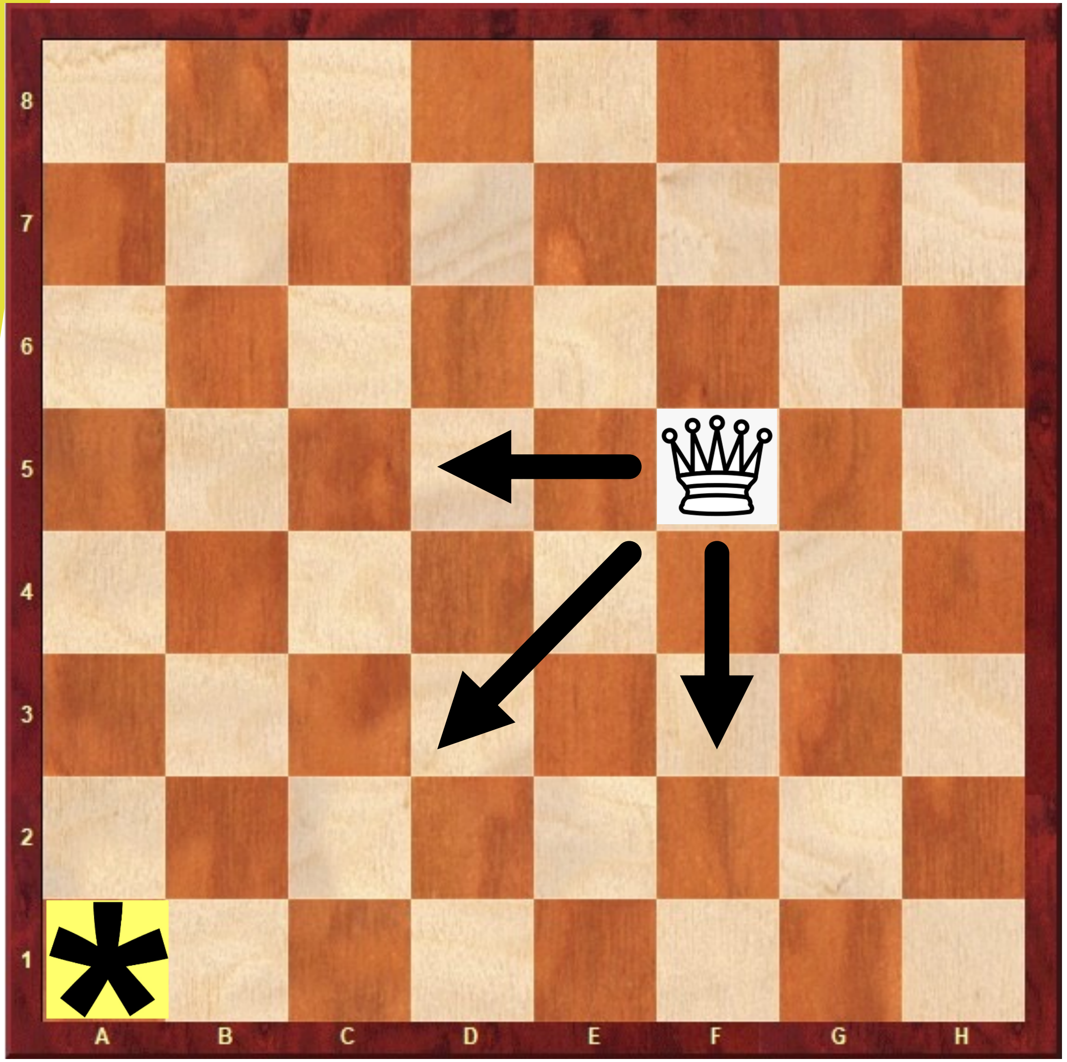
Game Analysis



All future positions can be enumerated

Players have the capacity to determine their next move without an element of luck

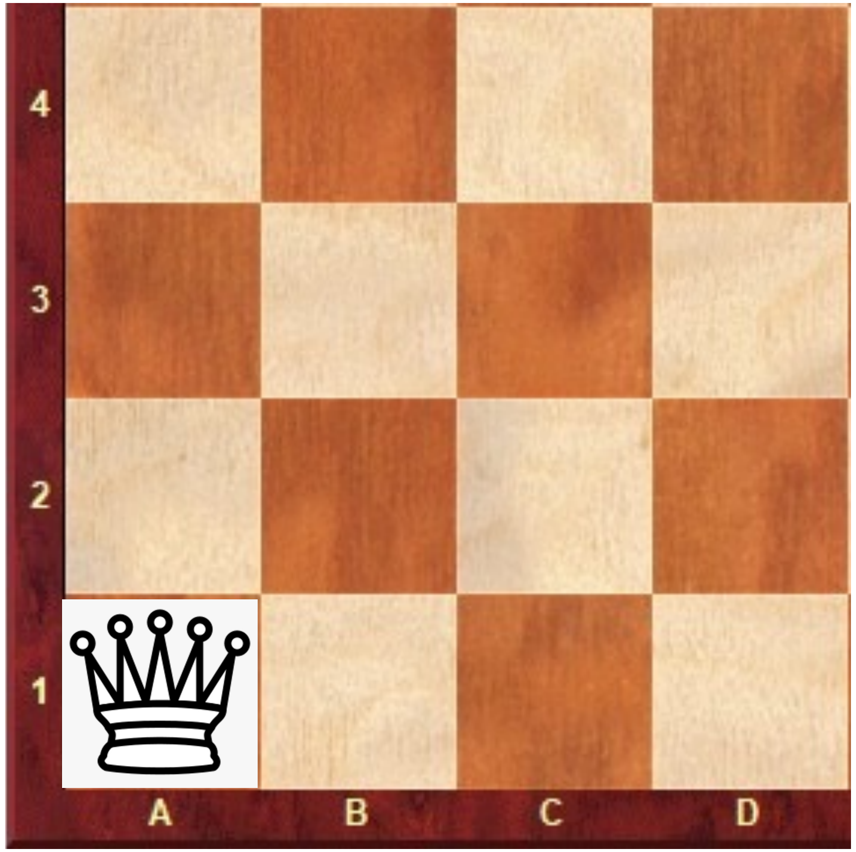
Game Analysis



Analyze the game by observing how game states build on each other

Let's build from simple cases.

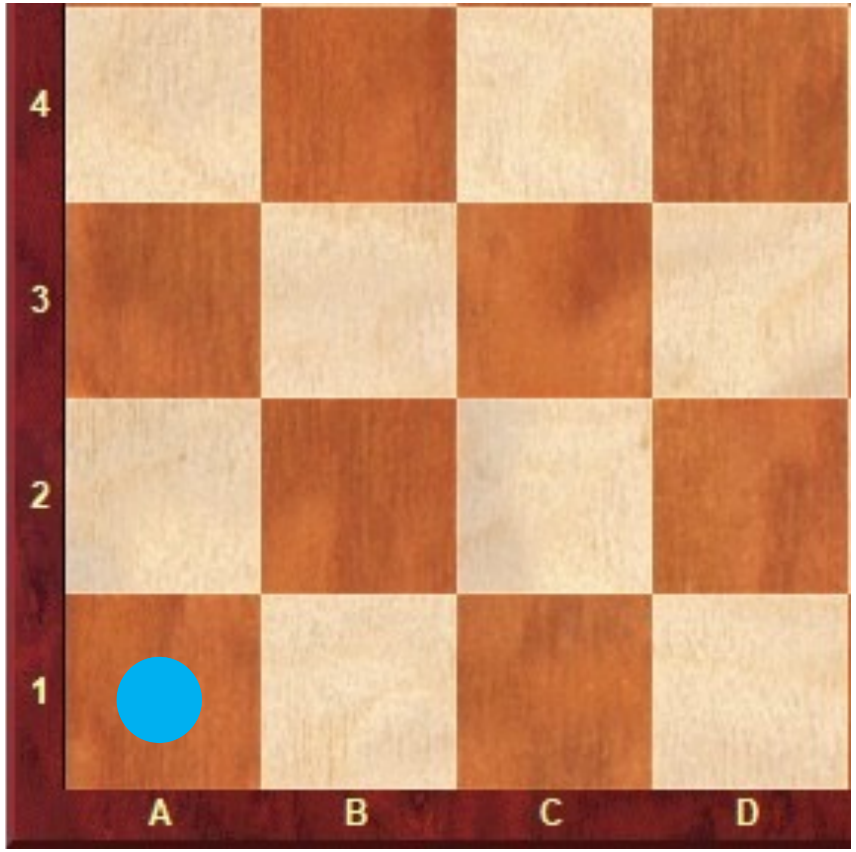
Game Analysis



Queen starts at A1

The second player wins because the first player is unable to move.

Game Analysis

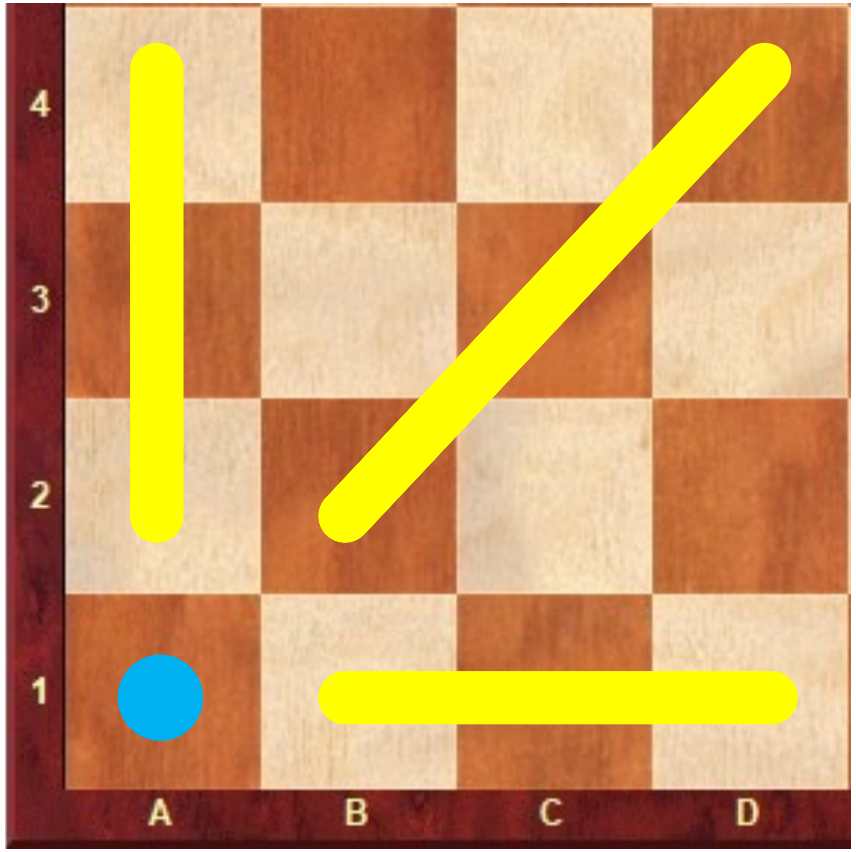


Queen starts at A1

Cold position

the player that starts in that position cannot win regardless of what moves he/she makes

Game Analysis



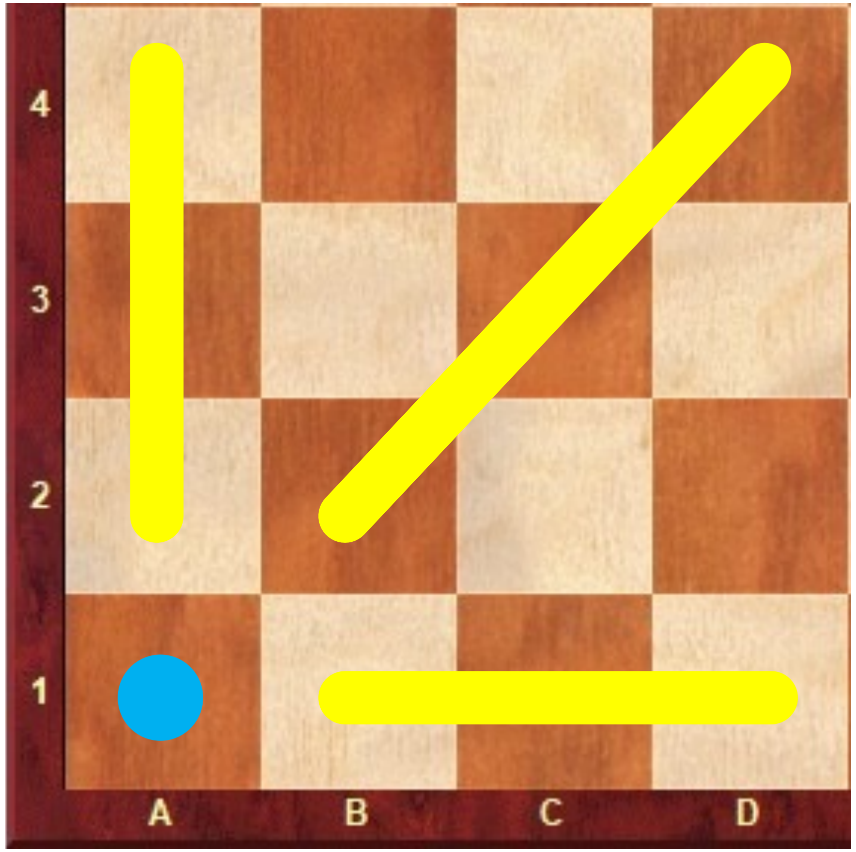
If the queen starts at a spot on a yellow line, the first player can win. He/she can move the queen to the **cold position**, and then the second player, who starts in that position, will lose.

Game Analysis

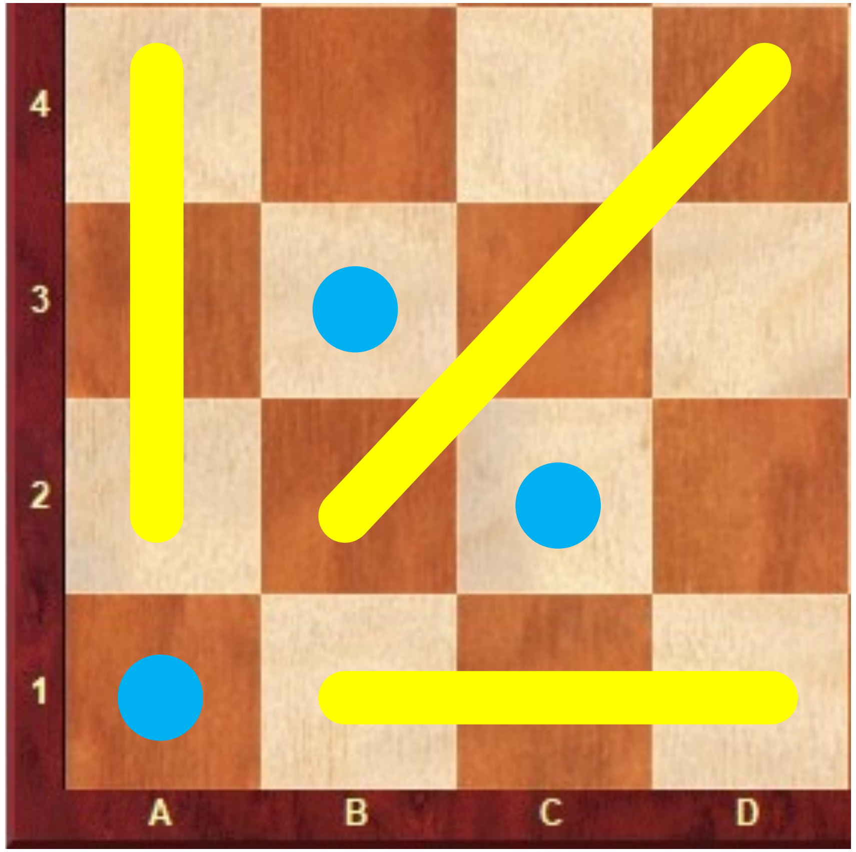
If the queen starts in a position that can be moved to a **cold position**, the first player will always win.
The first player can move to the **cold position**, and the second player, who has to move now, will lose.

We will call such positions **hot positions**.

Game Analysis



Game Analysis



B3 and C2 are **cold positions.**

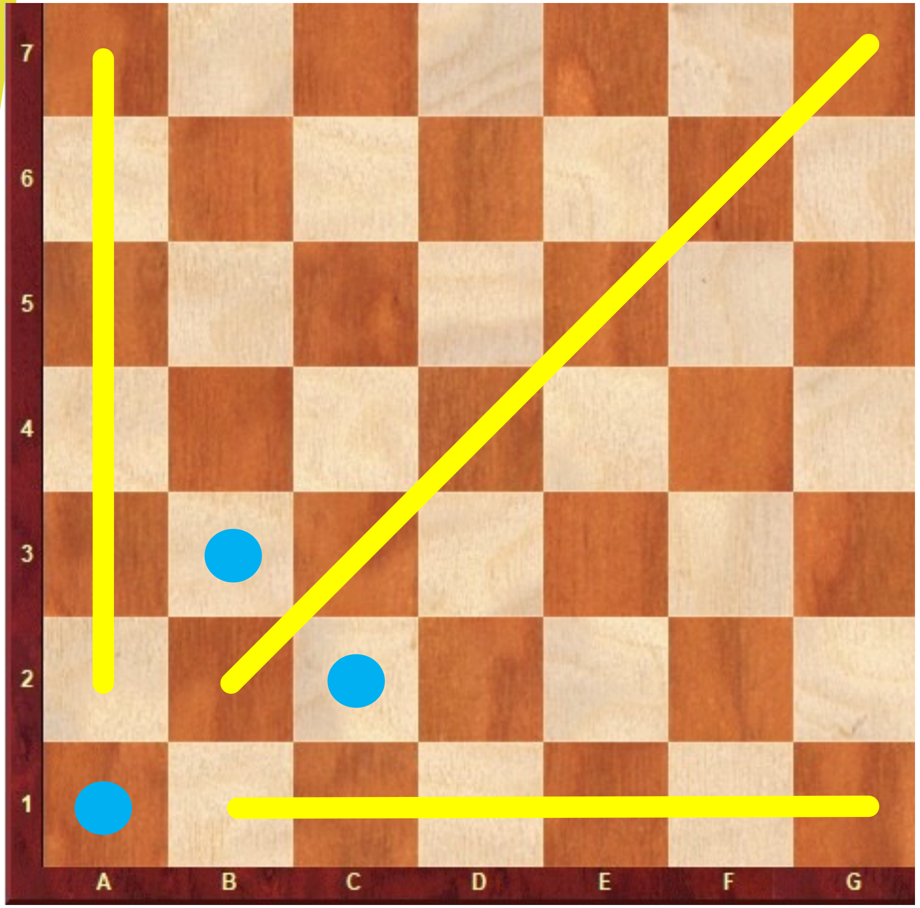
If the queen starts at these positions, every possible move is always to a **hot position**.

Game Analysis

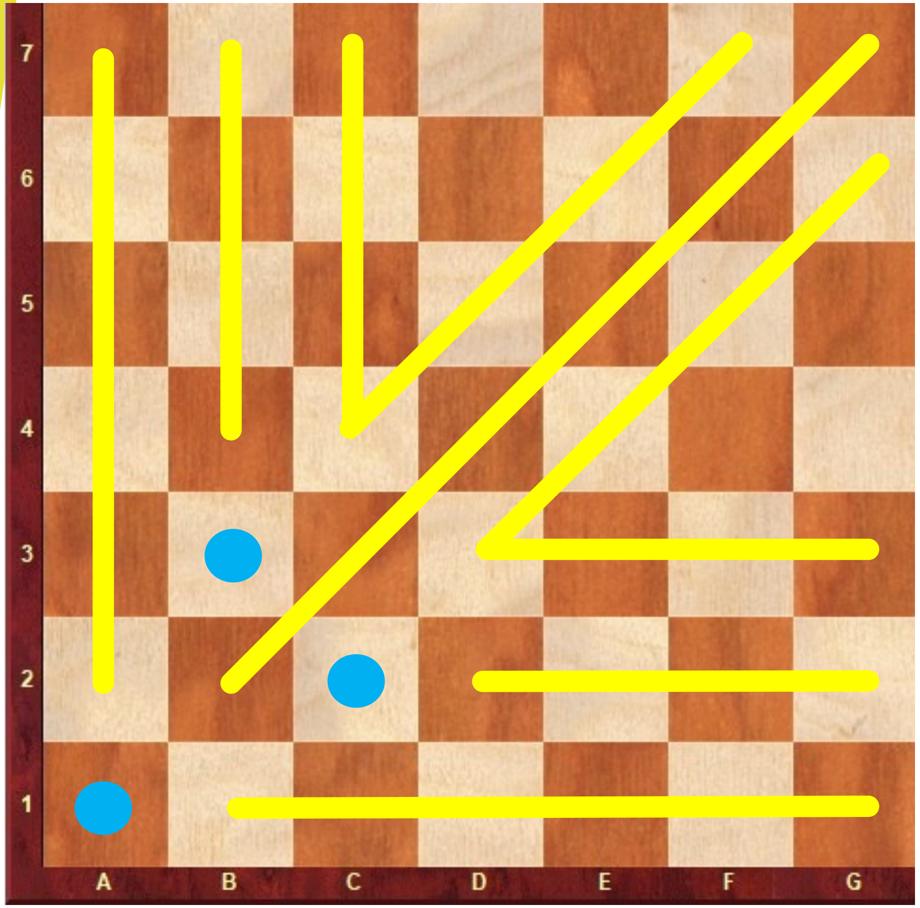
By definition, a **cold position** is a position where all moves are to **hot positions**.

Each position is a **cold position** or a **hot position**. It can't be neither.

Game Analysis

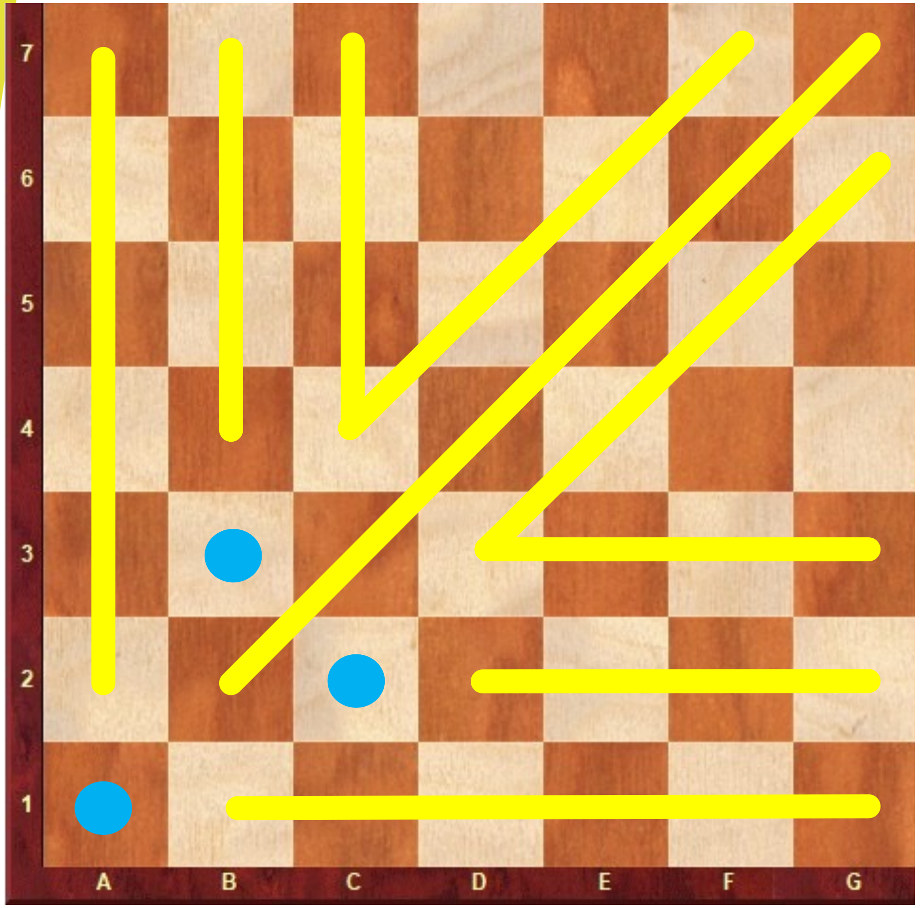


Game Analysis



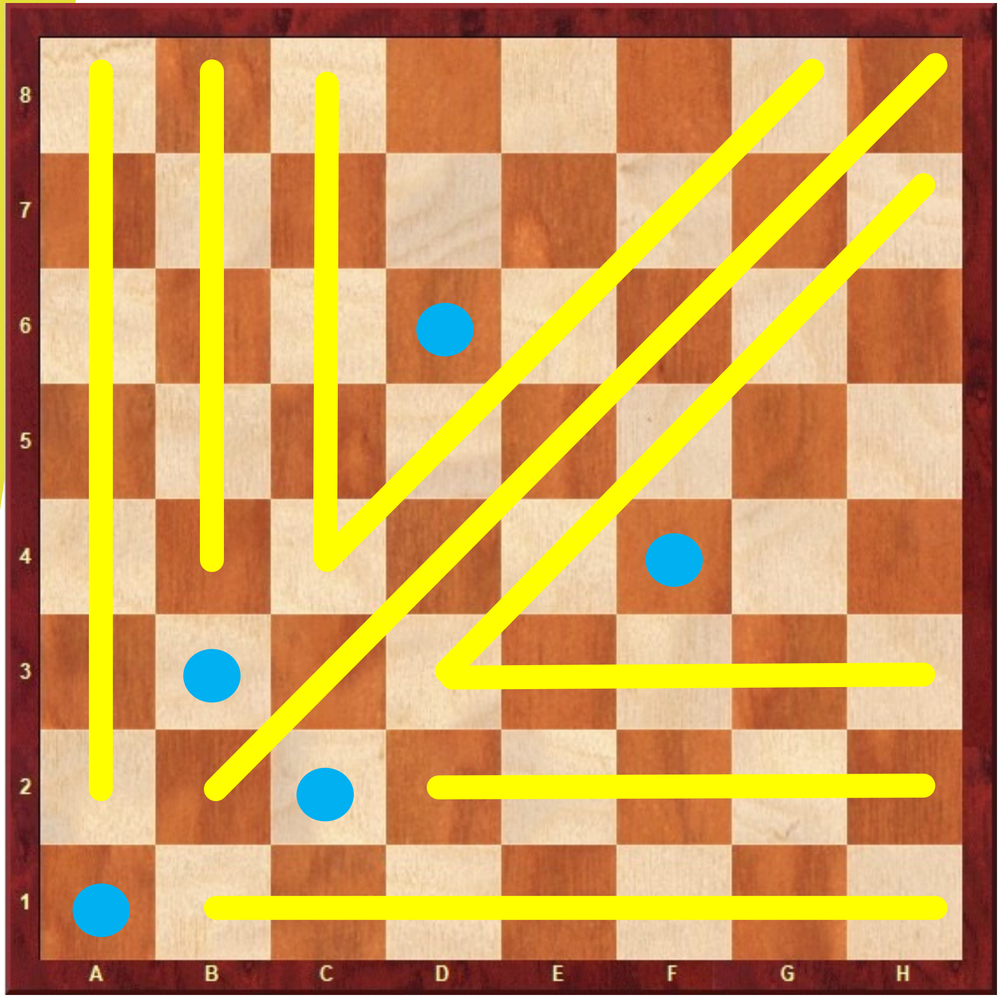
The positions from which a queen could be moved to B3 or C2 are **hot positions.**

Game Analysis



Repeating this process…

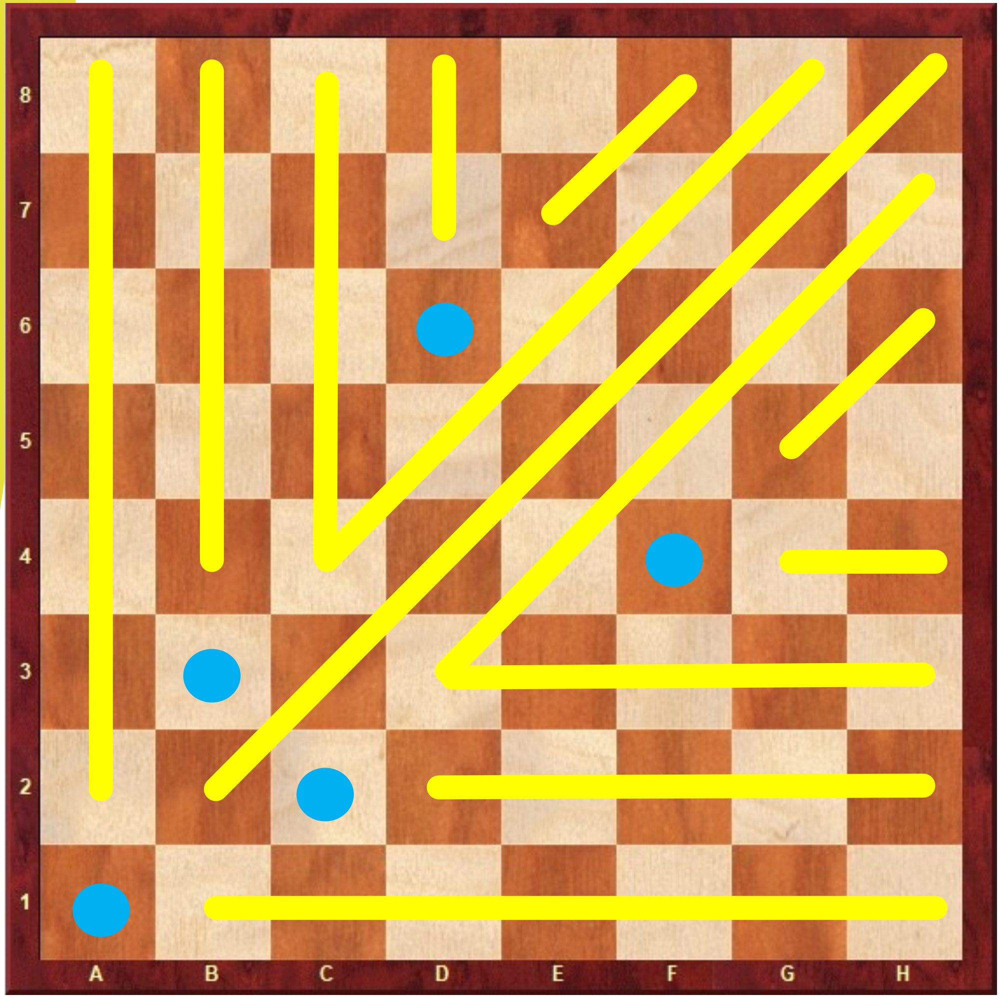
Game Analysis



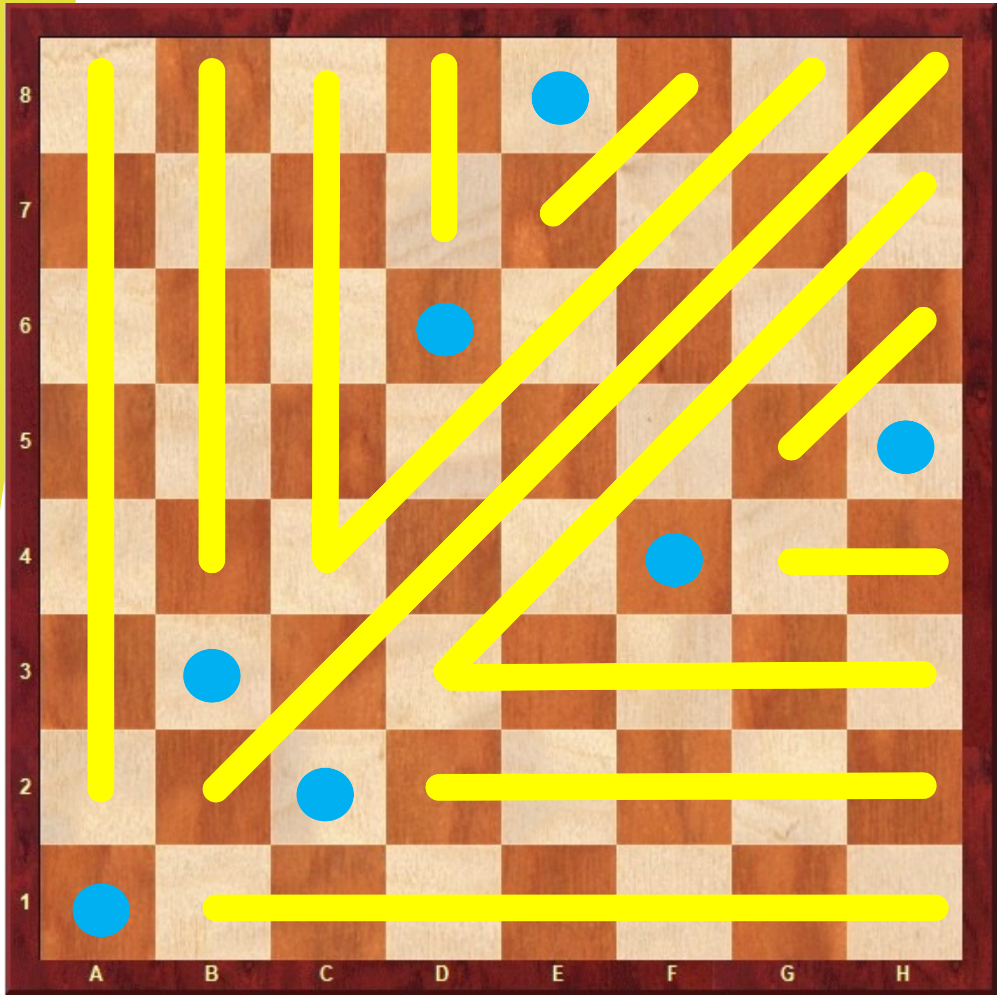
D6 and F4 are **cold positions.**

If the queen starts at these positions, every possible move is always to a **hot position.**

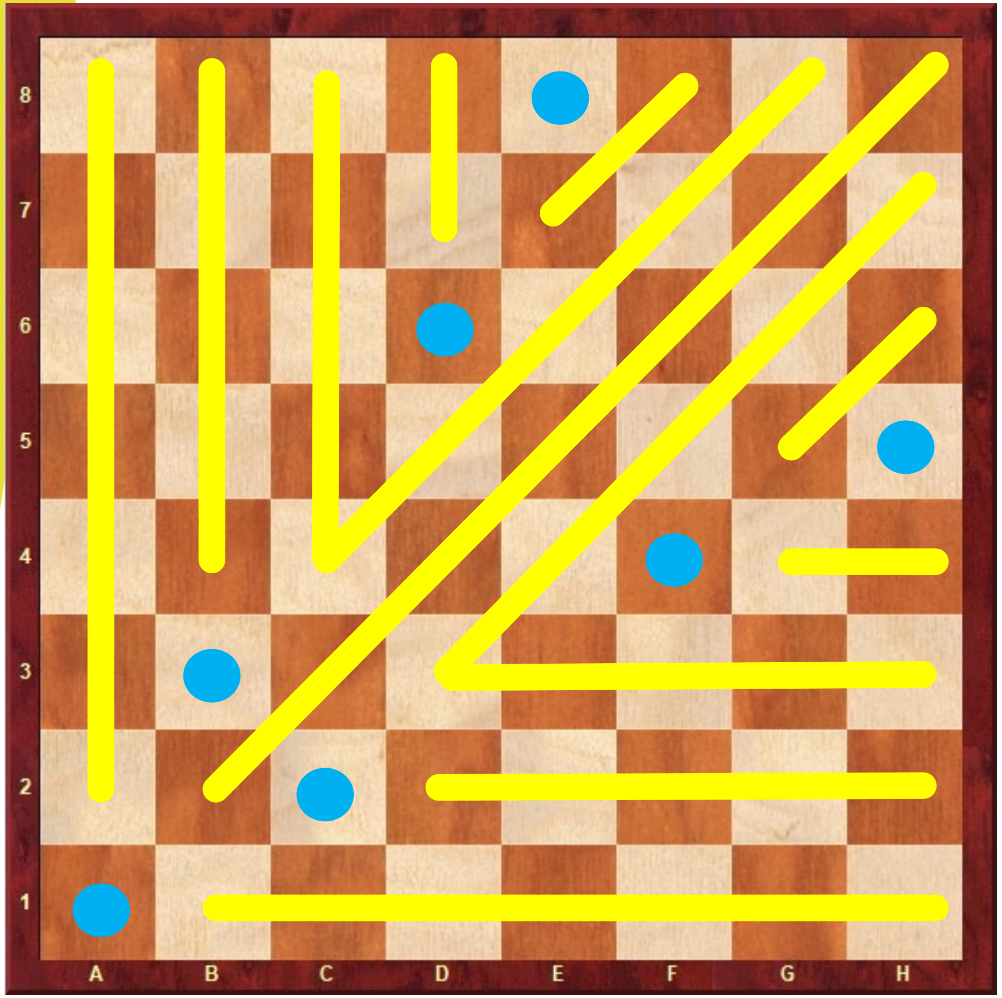
Game Analysis



Game Analysis

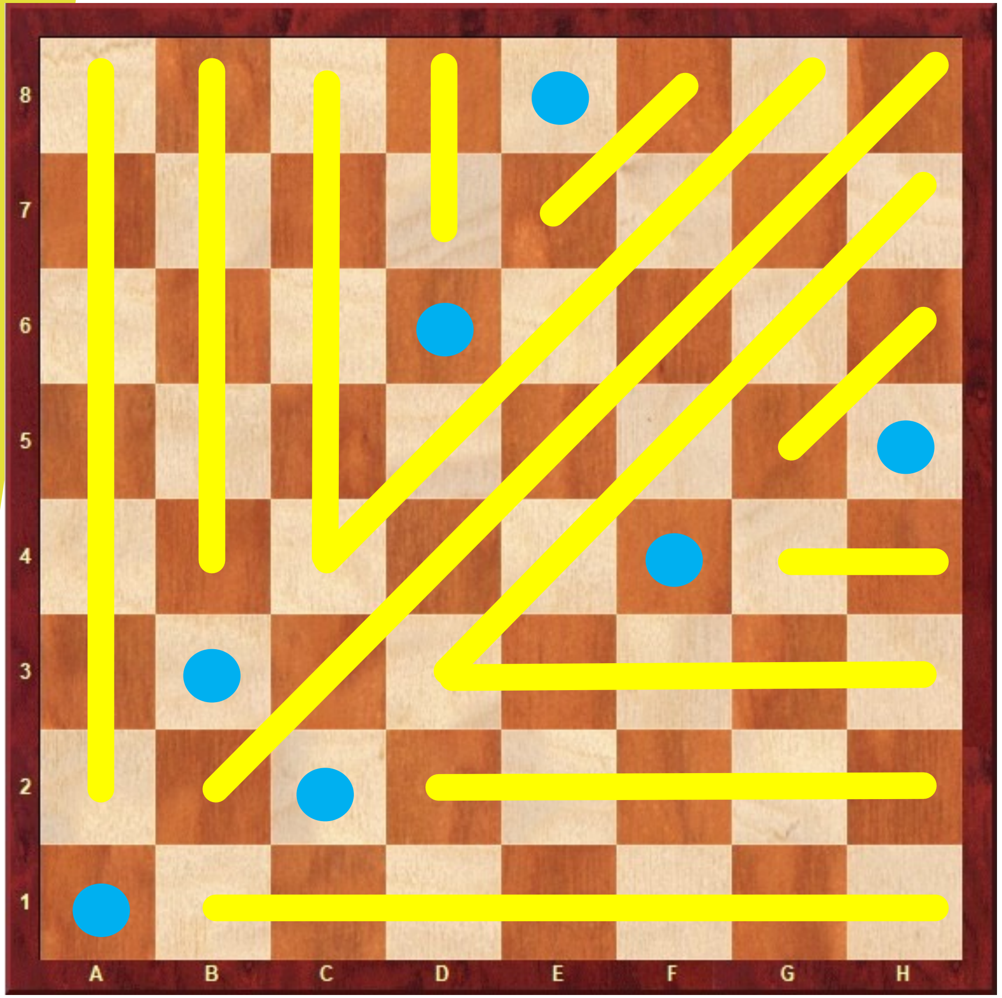


Game Strategy



If the game starts with the queen on a **hot position**, the first player should always move to a **cold position.**

Game Strategy



If the game starts with the queen on a **cold position**, the first player will lose, assuming the opponent plays optimally.

Better luck next time!

Interesting fact

Golden ratio?

Interesting fact

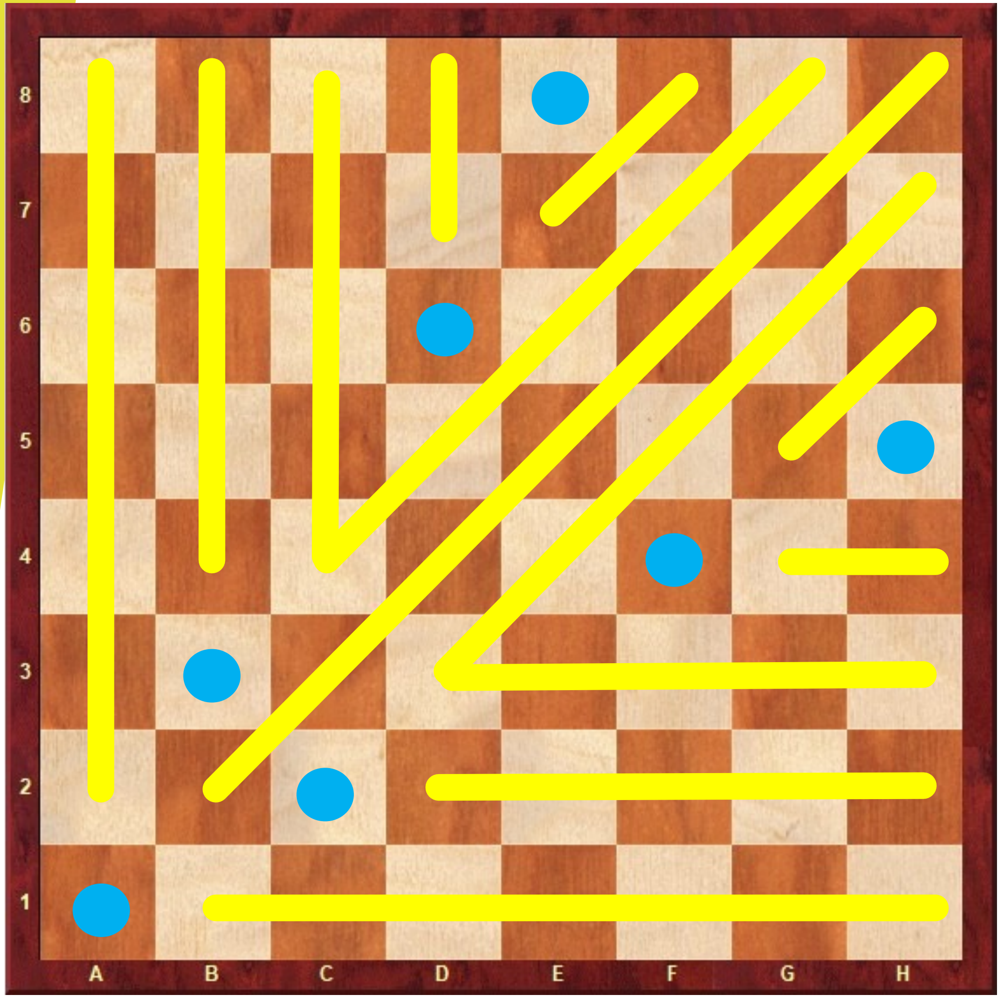
Golden ratio?

$$\phi = \frac{1+\sqrt{5}}{2}$$

Interesting fact

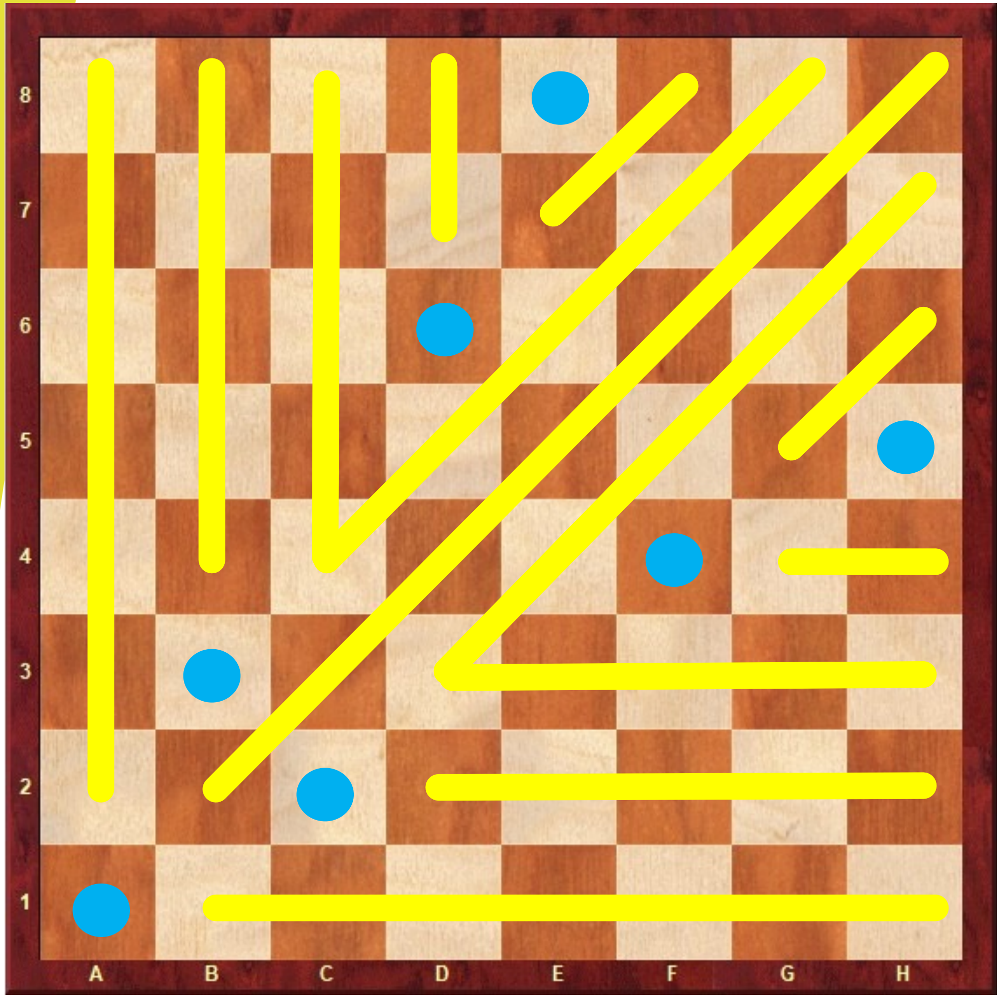
Every **cold position** in Wythoff's game can be computed using this ratio!

Interesting fact



Let's write the **cold positions** as ordered pairs of coordinates.

Interesting fact



(0, 0), (2, 1), (5, 3), (7, 4), …

(We will ignore the other positions because they are just reflections about the diagonal.)

Interesting fact

(0, 0), (2, 1),
(5, 3), (7, 4), …

Interesting fact

(0, 0), (2, 1), (5, 3), (7, 4), …

Interesting fact

(0, 0), (2, 1), (5, 3), (7, 4), …

Every pair is of the form

$$(\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)$$

where *n* is a non-negative integer {0, 1, 2, 3, …}.

Wythoff’s game

Thank you. Questions?