# Generating Functions

Ishwar Suriyaprakash

# Definition

The ”generating function” for a sequence $(a_1, a_2, a_3, a_4, \dots)$ is:

$$a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots$$

# Identities

Let the sequence

$$A = \left(\binom{n}{k}\right) = \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}\right).$$

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The generating function for *A* is

$$\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (x+1)^n$$

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Substituting 1 into *x*:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

# Identities

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (x+1)^n$$

Substituting -1 into *x*:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

# Identities

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (x+1)^n$$

Taking the derivative?

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Taking the derivative:

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Substituting 1 into *x*:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$$

# Combinatorics

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Given 2 6-sided dice, in how many ways can you roll a sum of 5?

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Dice 1: (1 OR 2 OR 3 OR 4 OR 5 OR 6)

AND

Dice 2: (1 OR 2 OR 3 OR 4 OR 5 OR 6)

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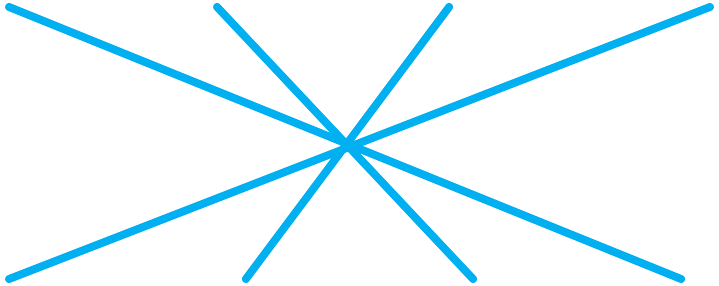

Dice 2: (1 OR 2 OR 3 OR 4 OR 5 OR 6)

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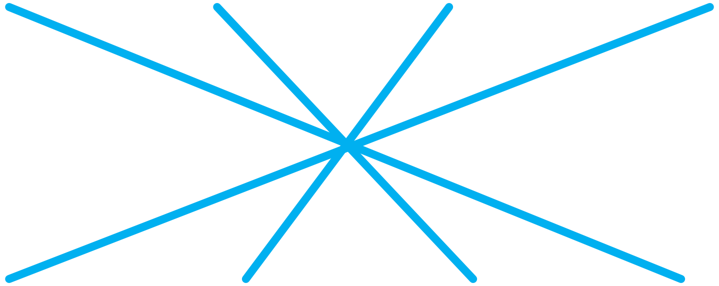

4 ways

Dice 2: (1 OR 2 OR 3 OR 4 OR 5 OR 6)

# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

Another way to represent the same question using polynomials …

# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

After multiplying the product

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

what is the coefficient of the term with $x^5$?

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$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8$$
$$4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

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# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

Poly 1: $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

$*$

Poly 2: $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

Finding the coefficient of $x^5$:

Poly 1: $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

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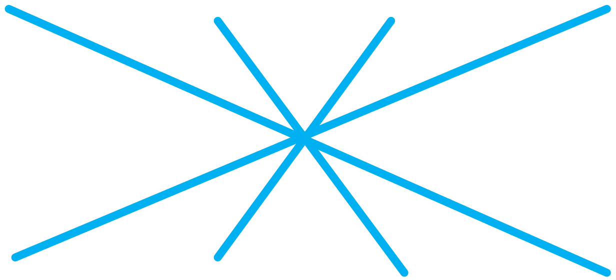

Poly 2: $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

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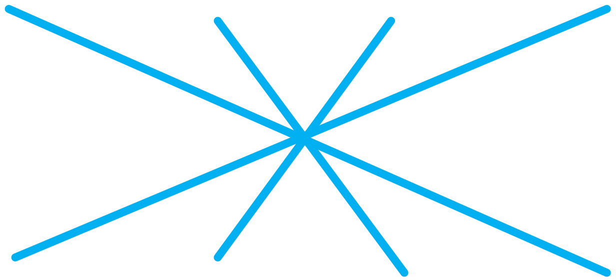

4 as coef.

Poly 2: $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

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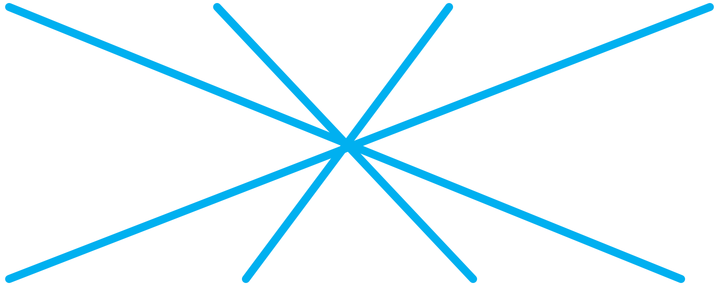

4 ways

Dice 2: (1 OR 2 OR 3 OR 4 OR 5 OR 6)

# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

$\downarrow\uparrow$

Find the coefficient of $x^5$ in the product:

$$(x^1+x^2+x^3+x^4+x^5+x^6)(x^1+x^2+x^3+x^4+x^5+x^6)$$

# Combinatorics

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

$\downarrow\uparrow$

Find the coefficient of $x^5$ in the product:

$$\underbrace{(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)}_{\text{Possibilities for Dice 1}}\underbrace{(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)}_{\text{Possibilities for Dice 2}}$$

# Partitions

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2 distinct partitions

# Partitions

- A *partition* of a positive integer $n$ is a representation of $n$ as a sum of other positive integers
- E.g. $5 = 1 + 4 = 1 + 2 + 2$

2 distinct partitions

Note: $1 + 4$ and $4 + 1$ are NOT distinct partitions.

# Partitions

Problem:

# Partitions

Problem: For any natural number *n*, prove that the number of partitions into unequal parts equals the number of partitions into odd parts.

**Unequal parts**: the numbers in each partition are distinct

**Odd parts**: the numbers in each partition are odd

# Partitions

Example (unequal parts):

$$5 = 5$$
$$= 4 + 1$$
$$= 3 + 2$$

# Partitions

Example (unequal parts):

$$8 = 8$$
$$= 7 + 1$$
$$= 6 + 2$$
$$= 5 + 3$$
$$= 5 + 2 + 1$$
$$= 4 + 3 + 1$$
~~= 3 + 3 + 1 + 1~~

# Partitions

Example (odd parts):

$$5 = 5$$
$$= 3 + 1 + 1$$
$$= 1 + 1 + 1 + 1 + 1$$

# Partitions

Example (odd parts):

$8 = 7 + 1$
$= 5 + 3$
$= 5 + 1 + 1 + 1$
$= 3 + 3 + 1 + 1$
$= 3 + 1 + 1 + 1 + 1 + 1$
$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
~~= 6 + 2~~

# Partitions

How do we prove this?

# Partitions

Given 2 6-sided dice, in how many ways can you roll a sum of 5?

$$\downarrow\uparrow$$

Find the coefficient of $x^5$ in the product:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

# Partitions

Let sequence $C = (c_0, c_1, c_2, \ldots)$, where $c_n$ is the number of partitions of a number $n$ into unequal parts.

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Let sequence $C = (c_0, c_1, c_2, \dots)$, where $c_n$ is the number of partitions of a number $n$ into unequal parts.

Let sequence $D = (d_0, d_1, d_2, \dots)$, where $d_n$ is the number of partitions of a number $n$ into odd parts.

# Partitions

Let sequence $C = (c_0, c_1, c_2, \dots)$, where $c_n$ is the number of partitions of a number $n$ into unequal parts.

Let sequence $D = (d_0, d_1, d_2, \dots)$, where $d_n$ is the number of partitions of a number $n$ into odd parts.

Prove that $c_n = d_n$ for all $n \geq 0$.

# Partitions

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Defining generating functions:

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$$P_c(x) = c_0 + c_1 x + c_2 x^2 + \ldots$$
$$P_d(x) = d_0 + d_1 x + d_2 x^2 + \ldots$$

# Partitions

Defining generating functions:

$$P_c(x) = c_0 + c_1x + c_2x^2 + \ldots$$
$$P_d(x) = d_0 + d_1x + d_2x^2 + \ldots$$

Proving $c_n = d_n$ for all $n \geq 0$.

$$\downarrow\uparrow$$

Proving $P_c(x) = P_d(x)$

# Partitions

# Partitions

Solving for $P_c(x)$

# Partitions

Solving for $P_c(x)$

Partitions into unequal parts

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| 1 | OR | no 1 |

AND

| | | |
|---|---|---|
| 2 | OR | no 2 |

AND

| | | |
|---|---|---|
| 3 | OR | no 3 |

AND

| | | |
|---|---|---|
| 4 | OR | no 4 |

AND

| | | |
|---|---|---|
| 5 | OR | no 5 |

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| 1 | OR | 0 |
| 2 | OR | 0 |
| 3 | OR | 0 |
| 4 | OR | 0 |
| 5 | OR | 0 |

5 =

# Partitions

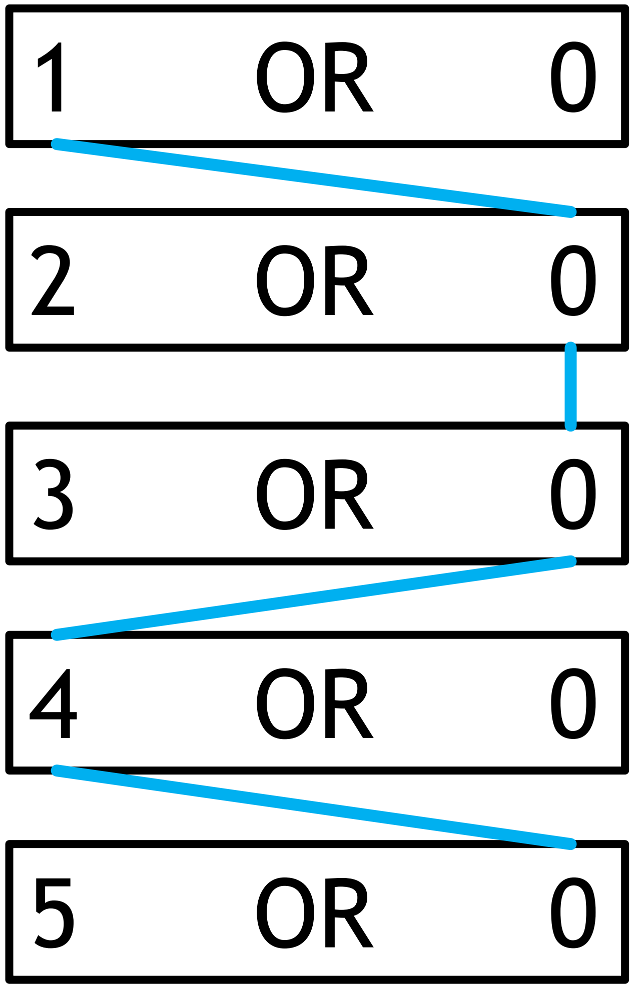
Solving for $P_c(x)$

| | | |
|---|---|---|
| 1 | OR | 0 |
| 2 | OR | 0 |
| 3 | OR | 0 |
| 4 | OR | 0 |
| 5 | OR | 0 |

5 = 5

# Partitions

Solving for $P_c(x)$




5 = 4 + 1

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| 1 | OR | 0 |
| 2 | OR | 0 |
| 3 | OR | 0 |
| 4 | OR | 0 |
| 5 | OR | 0 |

$5 = 3 + 2$

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| 1 | OR | 0 |

AND

| | | |
|---|---|---|
| 2 | OR | 0 |

AND

| | | |
|---|---|---|
| 3 | OR | 0 |

AND

| | | |
|---|---|---|
| 4 | OR | 0 |

AND

| | | |
|---|---|---|
| 5 | OR | 0 |

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| $x^1$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^2$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^3$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^4$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^5$ | + | $x^0$ |

# Partitions

Solving for $P_c(x)$

$$(x^1 + x^0) * (x^2 + x^0) * (x^3 + x^0) * (x^4 + x^0) * (x^5 + x^0)$$

# Partitions

Solving for $P_c(x)$

$$\left(x^1 + x^0\right) * \left(x^2 + x^0\right) * \left(x^3 + x^0\right) * \left(x^4 + x^0\right) * \left(x^5 + x^0\right)$$

# Partitions

Solving for $P_c(x)$

$$x^1 \quad + \quad x^0$$

$$*$$

$$x^2 \quad + \quad x^0$$

$$*$$

$$x^3 \quad + \quad x^0$$

$$*$$

$$x^4 \quad + \quad x^0$$

$$*$$

$$x^5 \quad + \quad x^0$$

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| $x^1$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^2$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^3$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^4$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^5$ | + | $x^0$ |

Distribution

$\rightarrow$ 3 $x^5$'s

# Partitions

Solving for $P_c(x)$

$$(x^1 + x^0) * (x^2 + x^0) * (x^3 + x^0) * (x^4 + x^0) * (x^5 + x^0)$$

Coefficient of $x^5$ is $c_5 = 3$

# Partitions

Solving for $P_c(x)$

$$x^1 + x^0$$

$$*$$

$$x^2 + x^0$$

$$*$$

$$x^3 + x^0$$

$$*$$

$$x^4 + x^0$$

$$*$$

$$\vdots$$

# Partitions

Solving for $P_c(x)$

| | | |
|---|---|---|
| $x^1$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^2$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^3$ | + | $x^0$ |

*

| | | |
|---|---|---|
| $x^4$ | + | $x^0$ |

*

.

.

.

Coefficient of $x^n$ is $c_n$

# Partitions

Solving for $P_c(x)$

In $(x^1 + x^0)(x^2 + x^0)(x^3 + x^0)$ …, the coefficient of $x^n$ is $c_n$.

# Partitions

Solving for $P_c(x)$

In $(x^1 + x^0)(x^2 + x^0)(x^3 + x^0)$ …, the coefficient of $x^n$ is $c_n$.

In $P_c(x)$, the coefficient of $x^n$ is $c_n$.

# Partitions

Solving for $P_c(x)$

So,

$$P_c(x) = (x^1 + x^0)(x^2 + x^0)(x^3 + x^0)\ldots$$

# Partitions

Solving for $P_d(x)$

# Partitions

Solving for $P_d(x)$

Partitions into odd parts

# Partitions

Solving for $P_d(x)$

| |
|---|
| no 1 OR one 1 OR two 1's OR … |

AND

| |
|---|
| no 3 OR one 3 OR two 3's OR … |

AND

| |
|---|
| no 5 OR one 5 OR two 5's OR … |

AND

| |
|---|
| no 7 OR one 7 OR two 7's OR … |

AND

| |
|---|
| no 9 OR one 9 OR two 9's OR … |

# Partitions

Solving for $P_d(x)$

0*1 OR one 1 OR two 1's OR ...

AND

0*3 OR one 3 OR two 3's OR ...

AND

0*5 OR one 5 OR two 5's OR ...

AND

0*7 OR one 7 OR two 7's OR ...

AND

0*9 OR one 9 OR two 9's OR ...

# Partitions

Solving for $P_d(x)$

| |
|---|
| 0*1 OR 1*1 OR two 1's OR … |

AND

| |
|---|
| 0*3 OR 1*3 OR two 3's OR … |

AND

| |
|---|
| 0*5 OR 1*5 OR two 5's OR … |

AND

| |
|---|
| 0*7 OR 1*7 OR two 7's OR … |

AND

| |
|---|
| 0*9 OR 1*9 OR two 9's OR … |

# Partitions

Solving for $P_d(x)$

| |
|---|
| 0*1 OR 1*1 OR 2*1 OR … |

AND

| |
|---|
| 0*3 OR 1*3 OR 2*3 OR … |

AND

| |
|---|
| 0*5 OR 1*5 OR 2*5 OR … |

AND

| |
|---|
| 0*7 OR 1*7 OR 2*7 OR … |

AND

| |
|---|
| 0*9 OR 1*9 OR 2*9 OR … |

# Partitions

Solving for $P_d(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*1 | OR | 1*1 | OR | 2*1 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*3 | OR | 1*3 | OR | 2*3 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*5 | OR | 1*5 | OR | 2*5 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*7 | OR | 1*7 | OR | 2*7 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*9 | OR | 1*9 | OR | 2*9 | OR | … |

# Partitions

Solving for $P_d(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*1 | OR | 1*1 | OR | 2*1 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*3 | OR | 1*3 | OR | 2*3 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*5 | OR | 1*5 | OR | 2*5 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*7 | OR | 1*7 | OR | 2*7 | OR | … |

AND

| | | | | | | |
|---|---|---|---|---|---|---|
| 0*9 | OR | 1*9 | OR | 2*9 | OR | … |

# Partitions

Solving for $P_d(x)$

| |
|---|
| 0*1 OR 1*1 OR 2*1 OR … |

AND

| |
|---|
| 0*3 OR 1*3 OR 2*3 OR … |

AND

| |
|---|
| 0*5 OR 1*5 OR 2*5 OR … |

AND

| |
|---|
| 0*7 OR 1*7 OR 2*7 OR … |

AND

| |
|---|
| 0*9 OR 1*9 OR 2*9 OR … |

5 =

# Partitions

Solving for $P_d(x)$

0*1 OR 1*1 OR 2*1 OR ...

0*3 OR 1*3 OR 2*3 OR ...

0*5 OR 1*5 OR 2*5 OR ...

0*7 OR 1*7 OR 2*7 OR ...

0*9 OR 1*9 OR 2*9 OR ...

5 = 5

# Partitions

Solving for $P_d(x)$

0*1 OR 1*1 OR 2*1 OR ...

0*3 OR 1*3 OR 2*3 OR ...

0*5 OR 1*5 OR 2*5 OR ...

0*7 OR 1*7 OR 2*7 OR ...

0*9 OR 1*9 OR 2*9 OR ...

5 = 3 + 1
+ 1

# Partitions

Solving for $P_d(x)$

| |
|---|
| 0*1 OR 1*1 OR 2*1 OR … |

AND

| |
|---|
| 0*3 OR 1*3 OR 2*3 OR … |

AND

| |
|---|
| 0*5 OR 1*5 OR 2*5 OR … |

AND

| |
|---|
| 0*7 OR 1*7 OR 2*7 OR … |

AND

| |
|---|
| 0*9 OR 1*9 OR 2*9 OR … |

# Partitions

## Solving for $P_d(x)$

$$x^0 + x^1 + x^2 + \ldots$$

$$*$$

$$x^0 + x^3 + x^6 + \ldots$$

$$*$$

$$x^0 + x^5 + x^{10} + \ldots$$

$$*$$

$$x^0 + x^7 + x^{14} + \ldots$$

$$*$$

$$x^0 + x^9 + x^{18} + \ldots$$

# Partitions

Solving for $P_d(x)$

$$x^0 + x^1 + x^2 + \ldots$$

$$*$$

$$x^0 + x^3 + x^6 + \ldots$$

$$*$$

$$x^0 + x^5 + x^{10} + \ldots$$

$$*$$

$$x^0 + x^7 + x^{14} + \ldots$$

$$*$$

$$x^0 + x^9 + x^{18} + \ldots$$

# Partitions

Solving for $P_d(x)$

$$x^0 + x^1 + x^2 + \ldots$$

$*$

$$x^0 + x^3 + x^6 + \ldots$$

$*$

$$x^0 + x^5 + x^{10} + \ldots$$

$*$

$$x^0 + x^7 + x^{14} + \ldots$$

$*$

$$x^0 + x^9 + x^{18} + \ldots$$

# Partitions

Solving for $P_d(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^1$ | + | $x^2$ | + | ... |

*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^3$ | + | $x^6$ | + | ... |

*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^5$ | + | $x^{10}$ | + | ... |

*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^7$ | + | $x^{14}$ | + | ... |

*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^9$ | + | $x^{18}$ | + | ... |

Coefficient of $x^5$ is $d_5 = 3$

# Partitions

Solving for $P_d(x)$

$$x^0 + x^1 + x^2 + \dots$$

$$*$$

$$x^0 + x^3 + x^6 + \dots$$

$$*$$

$$x^0 + x^5 + x^{10} + \dots$$

$$*$$

$$x^0 + x^7 + x^{14} + \dots$$

$$*$$

$$\vdots$$

# Partitions

Solving for $P_d(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^1$ | + | $x^2$ | + | … |

$*$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^3$ | + | $x^6$ | + | … |

$*$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^5$ | + | $x^{10}$ | + | … |

$*$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x^0$ | + | $x^7$ | + | $x^{14}$ | + | … |

$*$

.

.

Coefficient of $x^n$ is $d_n$

# Partitions

## Solving for $P_d(x)$

In $(x^0 + x^1 + x^2 + \ldots)(x^0 + x^3 + x^6 + \ldots)(x^0 + x^5 + x^{15} + \ldots)\ldots$

the coefficient of $x^n$ is $d_n$.

# Partitions

Solving for $P_d(x)$

In $(x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots) \dots$ the coefficient of $x^n$ is $d_n$.

In $P_d(x)$, the coefficient of $x^n$ is $d_n$.

# Partitions

Solving for $P_d(x)$

So,

$P_d(x)$ =

$$(x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots) \dots$$

# Partitions

Solving for $P_c(x)$

Partitions into unequal parts

$$P_c(x) = (x^1 + x^0)(x^2 + x^0)(x^3 + x^0) \ldots$$

# Partitions

Solving for $P_c(x)$

Partitions into unequal parts

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1) \ldots$$

# Partitions

Solving for $P_d(x)$

Partitions into odd parts

$P_d(x)$ =

$$(x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots)\dots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1) \ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1)\ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1+1)(x^2+1)(x^3+1)\ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1) \ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right) \ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1+1)(x^2+1)(x^3+1)\ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1+1)(x^2+1)(x^3+1)\ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1)\ldots$$
$$= \left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{1-x^8}{\cancel{1-x^4}}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{\cancel{1-x^6}}\right)\ldots$$

# Partitions

## Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1)\ldots$$

$$= \left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{\cancel{1-x^6}}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1+1)(x^2+1)(x^3+1)\ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right)\ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1 + 1)(x^2 + 1)(x^3 + 1) \ldots$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right) \ldots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = (x^1+1)(x^2+1)(x^3+1)\ldots$$

$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\left(\frac{1-x^8}{1-x^4}\right)\left(\frac{1-x^{10}}{1-x^5}\right)\left(\frac{1-x^{12}}{1-x^6}\right)\ldots$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots}$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots)\dots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots)\dots$$

Geometric series

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ ...)(x^0 + x^3 + x^6 + \ ...)(x^0 + x^5 + x^{15} + \ ...)\ ...$$

Geometric series:

$$r^0 + r^1 + r^2 + \ ...$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ldots)(x^0 + x^3 + x^6 + \ldots)(x^0 + x^5 + x^{15} + \ldots)\ldots$$

Geometric series:

$$r^0 + r^1 + r^2 + \ldots = \frac{1}{1-r}$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ldots)(x^0 + x^3 + x^6 + \ldots)(x^0 + x^5 + x^{15} + \ldots)\ldots$$

Geometric series:

$r^0 + r^1 + r^2 + \ldots = \frac{1}{1-r}$ when $|r| < 1$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ ...)(x^0 + x^3 + x^6 + \ ...)(x^0 + x^5 + x^{15} + \ ...)...$$

$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)...$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots)\dots$$

$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\dots$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_d(x) = (x^0 + x^1 + x^2 + \ \dots)(x^0 + x^3 + x^6 + \ \dots)(x^0 + x^5 + x^{15} + \ \dots)\dots$$

$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\dots$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\dots}$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots}$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = P_d(x) = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots}$$

# Partitions

Proving that $P_c(x) = P_d(x)$

$$P_c(x) = P_d(x)$$

# Partitions

Defining generating functions:

$$P_c(x) = c_0 + c_1 x + c_2 x^2 + \ldots$$
$$P_d(x) = d_0 + d_1 x + d_2 x^2 + \ldots$$

Proving $c_n = d_n$ for all $n \geq 0$.

$$\downarrow\uparrow$$

Proving $P_c(x) = P_d(x)$

# Partitions

Defining generating functions:

$$P_c(x) = c_0 + c_1 x + c_2 x^2 + \ \ldots$$
$$P_d(x) = d_0 + d_1 x + d_2 x^2 + \ \ldots$$

Proving $c_n = d_n$ for all $n \geq 0$.

$$\downarrow\uparrow$$

Proving $P_c(x) = P_d(x)$

# Partitions

Problem: For any natural number *n*, prove that the number of partitions into unequal parts equals the number of partitions into odd parts.

$\downarrow\uparrow$

Proving $c_n = d_n$ for all $n \geq 0$.

$\downarrow\uparrow$

Proving $P_c(x) = P_d(x)$

# Partitions

Problem: For any natural number *n*, prove that the number of partitions into unequal parts equals the number of partitions into odd parts.

$$\downarrow\uparrow$$

Proving $c_n = d_n$ for all $n \geq 0$.

$$\downarrow\uparrow$$

Proving $P_c(x) = P_d(x)$ ✓

# Partitions

Problem: For any natural number *n*, prove that the number of partitions into unequal parts equals the number of partitions into odd parts. ✓

$\downarrow\uparrow$

Proving $c_n = d_n$ for all $n \geq 0$. ✓

$\downarrow\uparrow$

Proving $P_c(x) = P_d(x)$ ✓

# Partitions

We proved it!

# References

# References

*The Art and Craft of Problem Solving*
Paul Zeitz