Catalan Objects

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Introduction

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Counting problems can have interesting answers

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A: F_n - the n^{th} Fibonacci number

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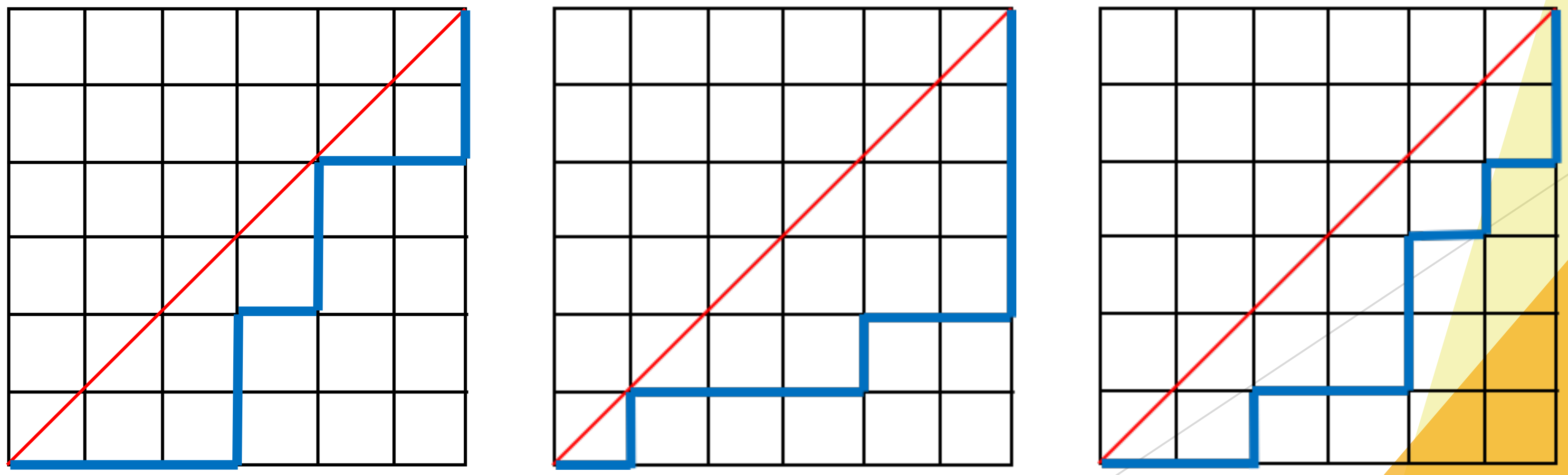
Example
Q: How many ways to tile a $1 \times n$ board with dominos (2x1) or squares (1x1)?
A: F_n - the n^{th} Fibonacci number

Sequences can have combinatorial interpretations

A Few Combinatorial Problems

1. Number of Dyck paths on an n x n grid
   Dyck path: path from $(0, 0)$ to (n, n) on grid such that
   - steps of length 1 either to the right or upwards on grid
   - each point on path not above line $y = x$

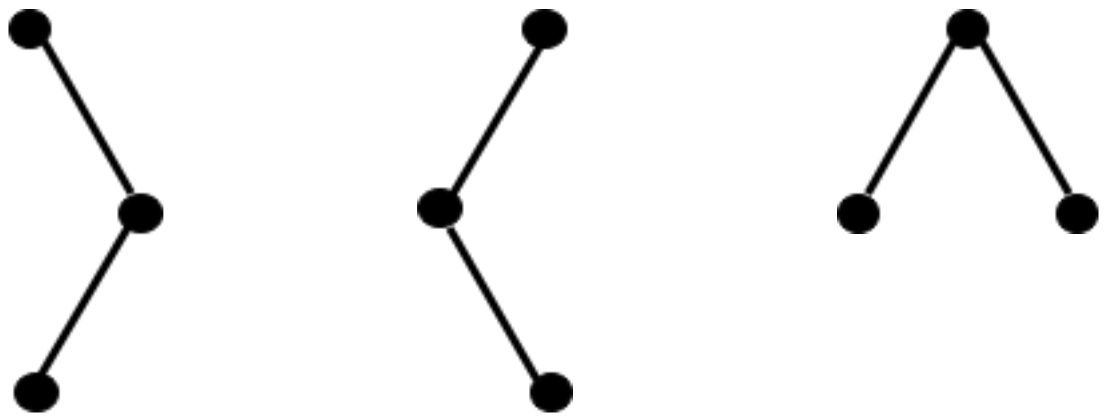
Example Dyck paths on a 6x6 grid



A Few Combinatorial Problems

1. Number of Dyck paths on an n x n grid
2. Number of unlabeled rooted binary trees with n vertices

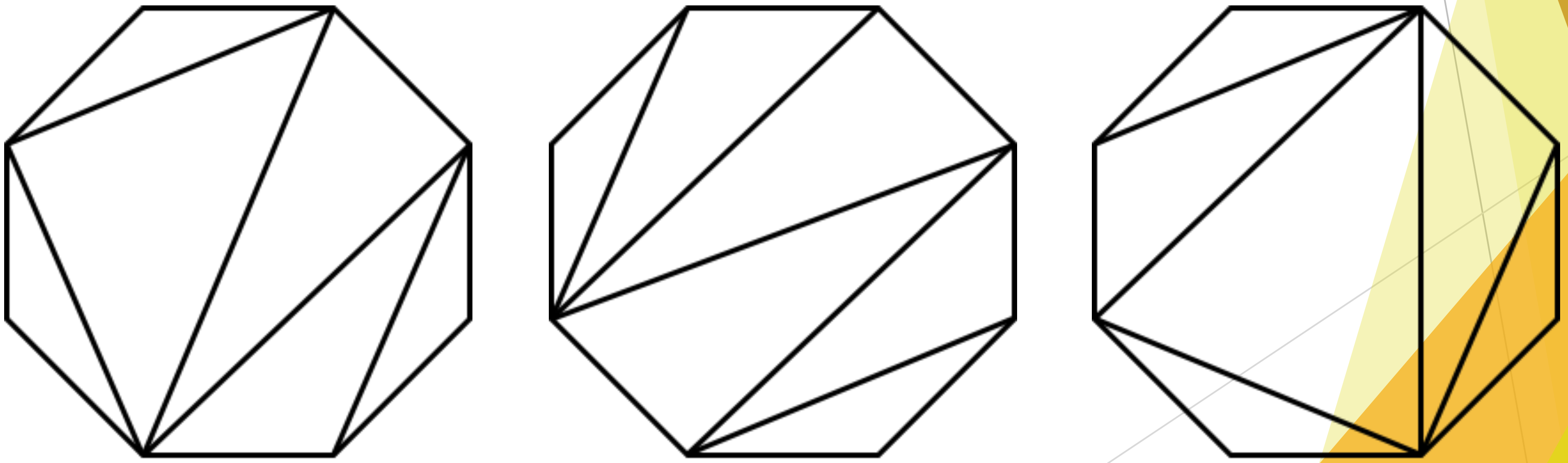
Example trees with 3 vertices



A Few Combinatorial Problems

1. Number of Dyck paths on an n x n grid
2. Number of unlabeled rooted binary trees with n vertices
3. Number of triangulations of a convex n-vertex polygon

Example triangulations of an octagon



A Few Combinatorial Problems

1. Number of Dyck paths on an n x n grid
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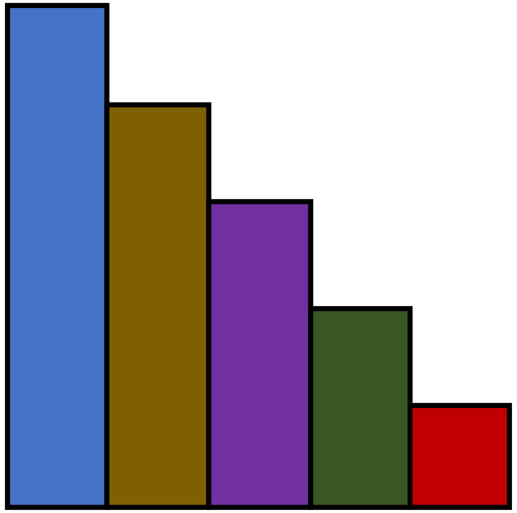
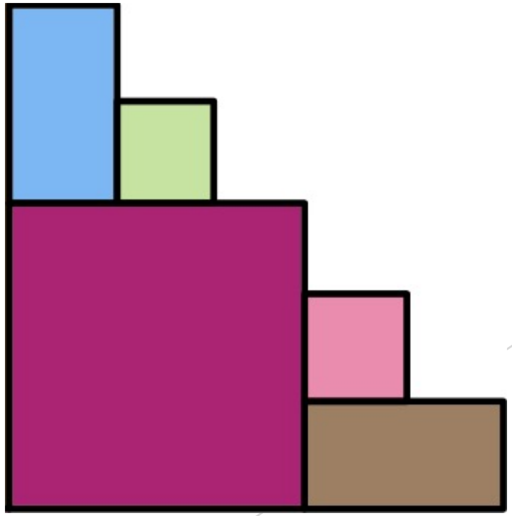
3. Number of triangulations of a convex n-vertex polygon
4. Number of tilings of a n-step staircase with rectangles

Example tilings of a 5-step staircase





A Few Combinatorial Problems

1. Number of Dyck paths on an n x n grid
2. Number of unlabeled rooted binary trees with n vertices
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Answer to each problem above is the following

$$\frac{1}{n+1}\binom{2n}{n} = \mathbf{C}_n, \textit{the } n^{th} \textit{ Catalan number}$$

This sequence is called the Catalan sequence

Revisiting Dyck Paths

1. Why is the number Dyck paths on an n x n grid $\mathbf{C}_n$?

Dyck path: path from $(0,0)$ to (n,n) on grid such that
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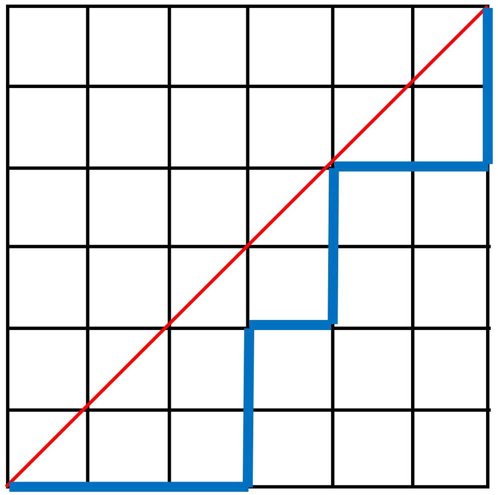
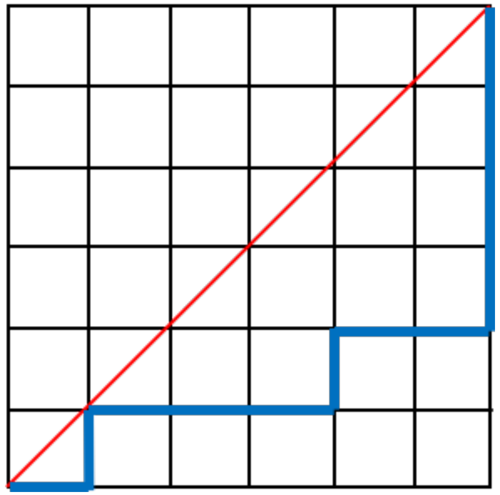
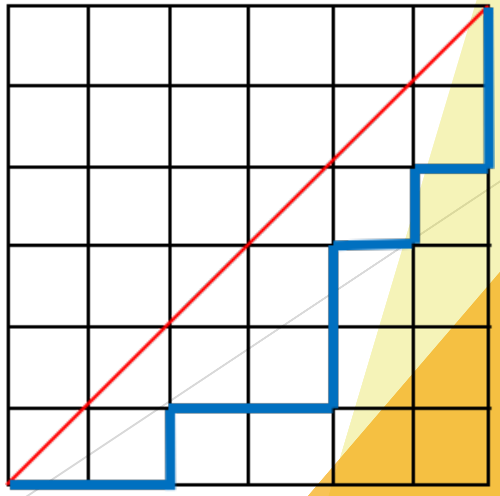
Example Dyck paths on a 6x6 grid

Revisiting Dyck Paths

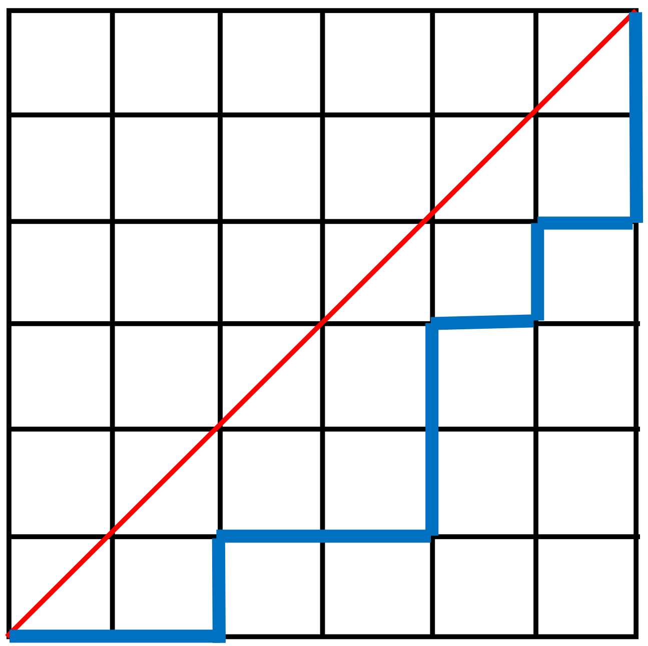
1. Why is the number Dyck paths on an n x n grid $\mathbf{C}_n$?

   Total number of paths from $(0, 0)$ to (n, n) is $\binom{2n}{n}$.

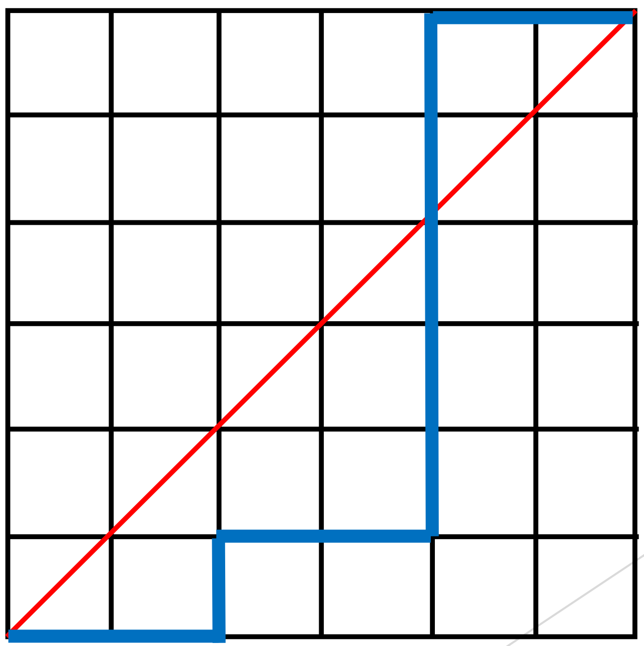
Revisiting Dyck Paths

1. Why is the number Dyck paths on an n x n grid $\mathbf{C}_n$?

Good Paths = $\mathbf{C}_n$



Bad Paths



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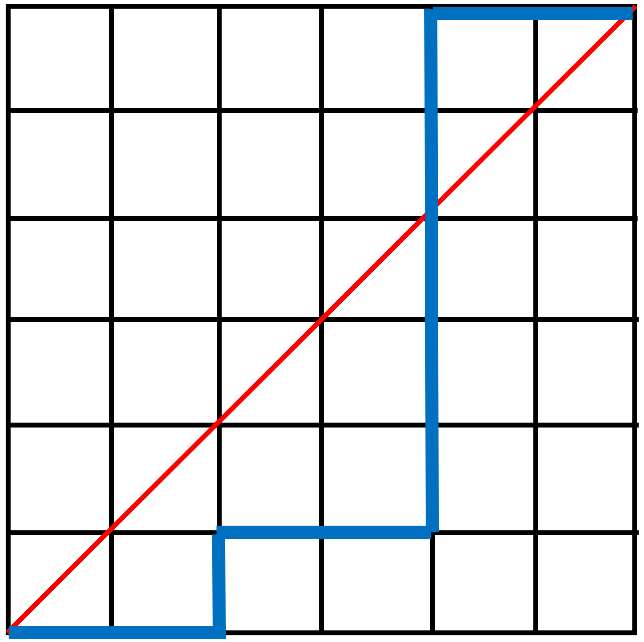

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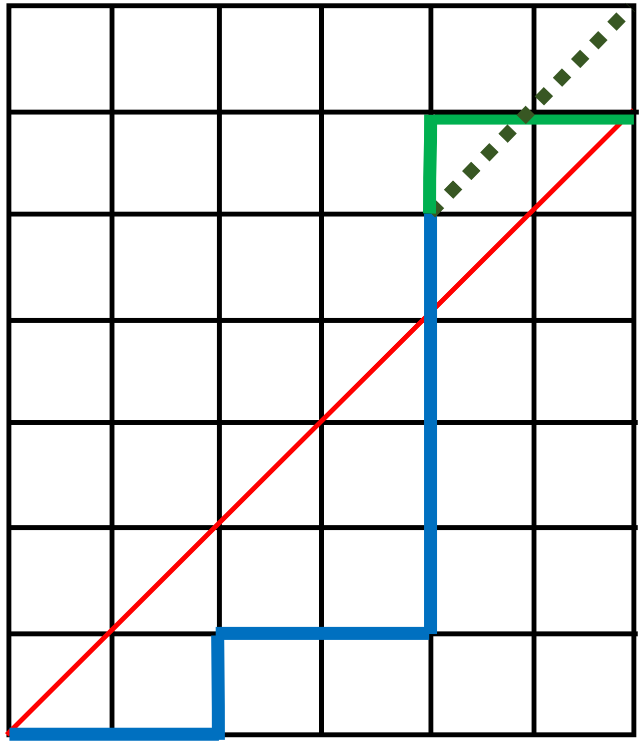

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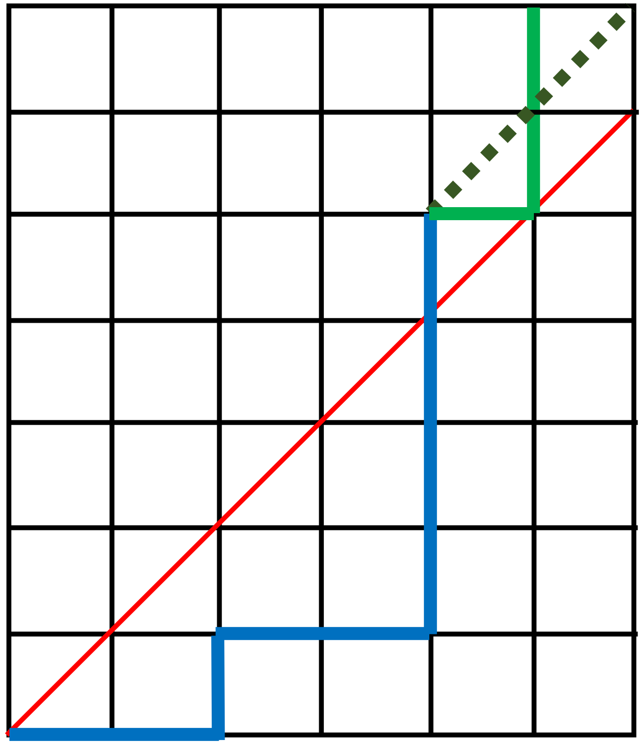
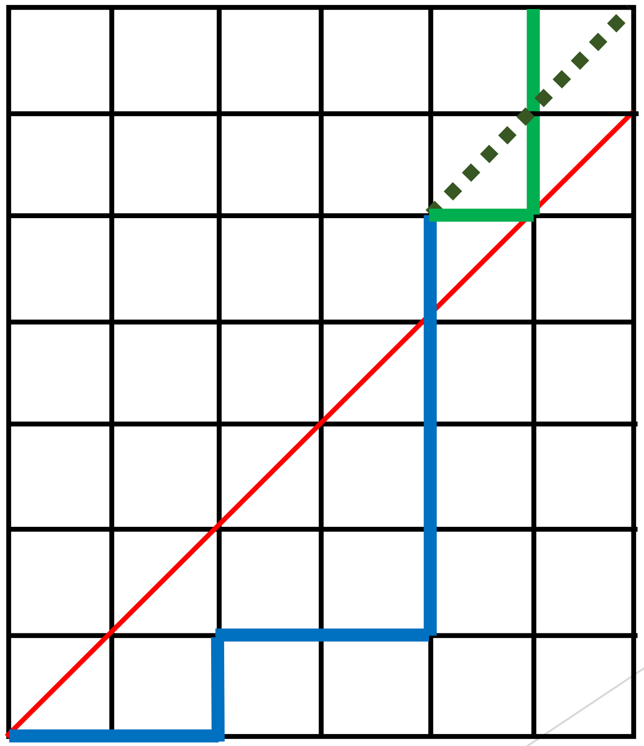


Revisiting Dyck Paths

1. Why is the number Dyck paths on an n x n grid $\mathbf{C}_n$?

Bad Paths

Number of paths from $(0,0)$ to $(n-1, n+1)$ is $\binom{2n}{n-1}$.



Revisiting Dyck Paths

1. Why is the number Dyck paths on an n x n grid $\mathbf{C}_n$?

   # of good paths = total - bad

$$= \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

$$= \mathbf{C}_n$$

Catalan Sequence & Recurrence

$$\frac{1}{n+1}\binom{2n}{n} = \mathbf{C}_n, \textit{the } n^{th} \textit{ Catalan number}$$

$\mathbf{C}_n$ can be represented as a recurrence relation

$$\mathbf{C}_n = \sum_{k=0}^{n-1} \mathbf{C}_k \mathbf{C}_{n-k-1}$$

$$= \mathbf{C}_0\mathbf{C}_{n-1} + \mathbf{C}_1\mathbf{C}_{n-2} + \ldots + \mathbf{C}_{n-2}\mathbf{C}_1 + \mathbf{C}_{n-1}\mathbf{C}_0$$

Revisiting Binary Trees

Number of unlabeled rooted binary trees with n vertices

Let B_n be the number of such trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$

Base cases: $B_0 = \mathbf{C}_0 = 1$, and $B_1 = \mathbf{C}_1 = 1$

Induction step: if $B_k = \mathbf{C}_k$ for all $k < n$, then $B_n = \mathbf{C}_n$

Binary Trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$

Constructing an n-vertex binary tree …

Binary Trees

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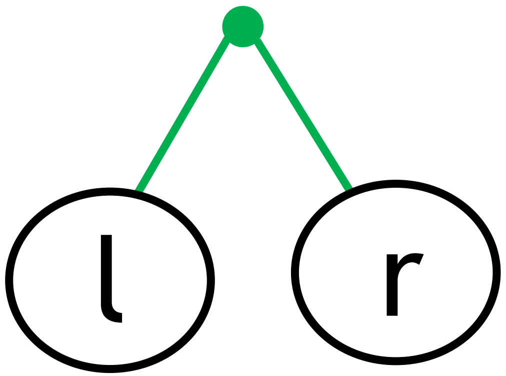
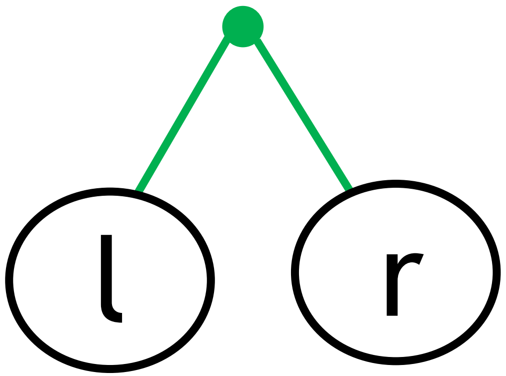


Conditions:

- l vertices on the left
- r vertices on the right
- $l + r + 1 = n$

Binary Trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$



B_l trees and B_r trees

Conditions:

- l vertices on the left
- r vertices on the right
- $l + r + 1 = n$

$\Rightarrow \quad B_l \times B_r$ trees

Binary Trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$

A binary tree with n vertices can have:

- 0 on left, n-1 on right
- 1 on left, n-2 on right
- .
- .
- .
- n-1 on left, 0 on right

Binary Trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$

Values for (l, r):

- 0 on left, n-1 on right $\Rightarrow B_0 \times B_{n-1}$
- 1 on left, n-2 on right $\Rightarrow B_1 \times B_{n-2}$
- .
- .
- .
- n-1 on left, 0 on right $\Rightarrow B_{n-1} \times B_0$

Binary Trees

Proof by induction to show that $B_n = \mathbf{C}_n$, for $n \geq 0$

$$B_n = B_0 B_{n-1} + B_1 B_{n-2} + \cdots + B_{n-1} B_0$$

$$= \sum_{k=0}^{n-1} B_k B_{n-k-1} = \mathbf{C}_n$$

Solution to Combinatorial Problems

1. Number of Dyck paths on an n x n grid
2. Number of unlabeled rooted binary trees with n vertices
3. Number of triangulations of a convex n-vertex polygon
4. Number of tilings of a n-step staircase with rectangles

Problems 3 and 4 can be either

a) Transformed to the Dyck paths problem, or

b) Proven to have the Catalan recurrence

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Problems 2, 3, and 4 can be either

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Not just these 4, about 214 combinatorial problems on graphs, strings, partitions, permutations have $\mathbf{C}_n$ as their answer!

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See *Catalan Numbers* by Richard P. Stanley for the 214 combinatorial problems